Problem 11757
(American Mathematical Monthly, Vol.121, February 2014)

Proposed by I. Gessel (USA).

Let $[x^a y^b]f(x,y)$ *denote the coefficient of* $x^a y^b$ *in the Taylor series expansion of* f*. Show that*

$$[x^n y^n]\frac{1}{(1-3x)(1-y-3x+3x^2)} = 9^n.$$

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Note that

$$\begin{aligned}
[y^n]\frac{1}{(1-3x)(1-y-3x+3x^2)} &= \frac{1}{(1-3x)(1-3x(1-x))^{n+1}} \\
&= \sum_{j=0}^{\infty}(3x)^j \sum_{k=0}^{\infty}\binom{n+k}{k}(3x(1-x))^k \\
&= \sum_{j=0}^{\infty}(3x)^j \sum_{k=0}^{\infty}\binom{n+k}{k}(3x)^k \sum_{i=0}^{k}\binom{k}{i}(-x)^i \\
&= \sum_{j=0}^{\infty}(3x)^j \sum_{m=0}^{\infty}x^m \sum_{k=0}^{m}\binom{n+k}{k}\binom{k}{m-k}3^k(-1)^{m-k}.
\end{aligned}$$

Hence

$$\begin{aligned}
[x^n y^n]\frac{1}{(1-3x)(1-y-3x+3x^2)} &= 3^n\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n+k}{k}\binom{k}{m-k}\left(-\frac{1}{3}\right)^{m-k} \\
&= 3^n\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n+m-k}{m-k}\binom{m-k}{k}\left(-\frac{1}{3}\right)^{k} \\
&= 3^n\sum_{k=0}^{n}\left(-\frac{1}{3}\right)^{k}\sum_{m=k}^{n}\binom{n+m-k}{m-k}\binom{m-k}{k} \\
&= 3^n\sum_{k=0}^{n}\left(-\frac{1}{3}\right)^{k}\binom{n+k}{k}\sum_{m=k}^{n}\binom{n+m-k}{n+k} \\
&= 3^n\sum_{k=0}^{\lfloor n/2\rfloor}\left(-\frac{1}{3}\right)^{k}\binom{n+k}{k}\binom{2n+1-k}{n+k+1}.
\end{aligned}$$

Thus, it suffices to show that

$$S(n) := \sum_{k=0}^{\lfloor n/2\rfloor}F(n,k) = 1 \qquad \text{where} \qquad F(n,k) := \frac{(-1)^k}{3^{n+k}}\binom{n+k}{k}\binom{2n+1-k}{n+k+1}.$$

Following the Wilf-Zeilberger method, let

$$G(n,k) := \frac{k(k-2(n+1))}{(n+1)(n+1-2k)}\cdot F(n,k)$$

then

$$F(n+1,k)-F(n,k) = G(n,k+1)-G(n,k) \qquad \text{for } 0 \le k < (n-1)/2.$$

Hence, if n is even then

$$S(n+1) - S(n) = \sum_{k=0}^{n/2} (F(n+1,k) - F(n,k)) = G(n, n/2+1) - G(n,0) = 0,$$

and, if n is odd,

$$\begin{aligned} S(n+1) - S(n) &= F(n+1,(n+1)/2) + F(n+1,(n-1)/2) - F(n,(n-1)/2) \\ &\quad + \sum_{k=0}^{(n-3)/2} (F(n+1,k) - F(n,k)) \\ &= F(n+1,(n+1)/2) + F(n+1,(n-1)/2) - F(n,(n-1)/2) \\ &\quad + G(n,(n-1)/2) - G(n,0) = 0. \end{aligned}$$

Since $S(0) = 1$, the identity follows by induction on n. $\square$