Problem 11756
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Proposed by P. Perfetti (Italy).

Let f be a function from $[1,1]$ to $\mathbb{R}$ with continuous derivatives of all orders up to $2n+2$. Given $f(0)=f''(0)=\cdots=f^{(2n+2)}(0)=0$, prove

$$\frac{(4n+5)((2n+2)!)^2}{2}\left(\int_{-1}^{1} f(x)\,dx\right)^2 \leq \int_{-1}^{1}\left(f^{(2n+2)}(x)\right)^2\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By the integral form of the remainder in the Taylor's Theorem we have that if $x\in[-1,1]$ then $f(x)=P_{2n}(x)+R_{2n}(x)$ where

$$P_{2n}(x)=\sum_{k=0}^{2n}\frac{f^{(k)}(0)}{k!}\,x^k \qquad \text{and} \qquad R_{2n}(x)=\int_0^x \frac{f^{(2n+2)}(t)}{(2n+1)!}\,(x-t)^{2n+1}\,dt.$$

Since $f(0)=f''(0)=\cdots=f^{(2n+2)}(0)=0$, it follows that P_{2n} is an odd function and

$$\int_{-1}^{1} P_{2n}(x)\,dx=0.$$

Therefore

$$\begin{aligned}
\int_{-1}^{1} f(x)\,dx &= \int_{-1}^{1} R_{2n}(x)\,dx\\
&= \int_{x=-1}^{0}\int_{t=x}^{0}\frac{f^{(2n+2)}(t)}{(2n+1)!}\,(t-x)^{2n+1}\,dt\,dx+\int_{x=0}^{1}\int_{t=0}^{x}\frac{f^{(2n+2)}(t)}{(2n+1)!}\,(x-t)^{2n+1}\,dt\,dx\\
&= \int_{t=-1}^{0}\int_{x=-1}^{t}\frac{f^{(2n+2)}(t)}{(2n+1)!}\,(t-x)^{2n+1}\,dx\,dt+\int_{t=0}^{1}\int_{x=t}^{1}\frac{f^{(2n+2)}(t)}{(2n+1)!}\,(x-t)^{2n+1}\,dx\,dt\\
&= \int_{t=-1}^{0}\frac{f^{(2n+2)}(t)}{(2n+1)!}\left[-\frac{(t-x)^{2n+2}}{2n+2}\right]_{x=-1}^{t}dt+\int_{t=0}^{1}\frac{f^{(2n+2)}(t)}{(2n+1)!}\left[\frac{(x-t)^{2n+2}}{2n+2}\right]_{x=t}^{1}dt\\
&= \int_{-1}^{0}\frac{f^{(2n+2)}(t)}{(2n+2)!}\,(t+1)^{2n+2}\,dt+\int_{0}^{1}\frac{f^{(2n+2)}(t)}{(2n+2)!}\,(1-t)^{2n+2}\,dt\\
&= \int_{-1}^{1}\frac{f^{(2n+2)}(t)}{(2n+2)!}\,h(t)\,dt,
\end{aligned}$$

where

$$h(t)=\begin{cases}(t+1)^{2n+2} & \text{if } t\in[-1,0]\\ (1-t)^{2n+2} & \text{if } t\in[0,1]\end{cases}.$$

Hence, by the Cauchy-Schwarz inequality,

$$\left(\int_{-1}^{1} f(x)\,dx\right)^2 \leq \int_{-1}^{1}(h(t))^2\,dt\cdot\int_{-1}^{1}\frac{(f^{(2n+2)}(t))^2}{((2n+2)!)^2}\,dt=\frac{2}{4n+5}\cdot\frac{1}{((2n+2)!)^2}\int_{-1}^{1}\left(f^{(2n+2)}(t)\right)^2\,dt$$

which is equivalent to the desired inequality. □