Problem 11755
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Proposed by P. P. Dályay (Hungary).

Compute

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{2k-1}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is well known that

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}=\frac{\pi}{4}\qquad\text{and}\qquad\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{8}.$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{2k-1}&=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\frac{\pi}{4}+\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\right)\\&=-\frac{\pi^2}{16}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\\&=-\frac{\pi^2}{16}+\sum_{1\leq k\leq n}\frac{(-1)^{n+k}}{(2k-1)(2n-1)}\\&=-\frac{\pi^2}{16}+\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}+\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\right)\\&=-\frac{\pi^2}{16}+\frac{1}{2}\left(\frac{\pi^2}{16}+\frac{\pi^2}{8}\right)=\frac{\pi^2}{32}.\end{aligned}$$

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