Problem 11754
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Proposed by D. Beckwith (USA).

When a fair coin is tossed n times, let $P(n)$ be the probability that the lengths of all runs (maximal constant strings) in the resulting sequence are of the same parity as n. Prove that

$$P(n) = \begin{cases} (1/2)^{n/2} & \text{if } n \text{ is even,} \\ (1/2)^{n-1} F_n & \text{if } n \text{ is odd,} \end{cases}$$

where F_n is the nth Fibonacci number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Assume first that n is even. Then the strings which start with Head and have d runs all of even lengths are counted by the solutions of the equation

$$2x_1 + 2x_2 + \cdots + 2x_d = n$$

where each x_i is a positive integer. The number of such solutions is $\binom{n/2-1}{d}$ and thus

$$P(n) = \frac{2}{2^n} \sum_{d=1}^{n/2} \binom{n/2-1}{d-1} = \frac{2^{n/2-1}}{2^{n-1}} = (1/2)^{n/2}.$$

Assume now that n is odd. Then the strings which start with Head and have d runs all of odd lengths are counted by the solutions of the equation

$$(2x_1 - 1) + (2x_2 - 1) + \cdots + (2x_d - 1) = n$$

where each x_i is a positive integer. The number of such solutions is $\binom{(n+d)/2-1}{d-1}$ when d is odd and it is zero otherwise. Thus, by setting $d = 2j + 1$, we obtain

$$P(n) = \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{(n+(2j+1))/2-1}{2j} = \frac{1}{2^{n-1}} \sum_{j=0}^{(n-1)/2} \binom{(n-1)/2+j}{2j} = (1/2)^{n-1} F_n,$$

where the last equality is due to the following fact. If $A(t) = \sum_{k\geq 0} a_k t^k$ then

$$\sum_{j=0}^{m} \binom{m+j}{2j} a_k = [t^m] \frac{1}{1-t} A\left(\frac{t}{(1-t)^2}\right).$$

Hence for $A(t) = 1/(1-t)$, we obtain

$$\sum_{j=0}^{m} \binom{m+j}{2j} = [t^m] \frac{1-t}{1-3t+t^2} = [t^{2m}] \frac{F(t) - F(-t)}{2t} = F_{2m+1}.$$

where $F(t) = 1/(1-t-t^2) = \sum_{k\geq 0} F_k t^k$. $\square$