Problem 11753
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Let f be a continuous map from $[0,1]$ to $\mathbb{R}$ that is differentiable on $(0,1)$, with $f(0)=0$ and $f(1)=1$. Show that for each positive integer n there exist distinct numbers $c_1,\ldots,c_n$ in $(0,1)$ such that

$$\prod_{k=1}^{n} f'(c_k)=1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Assume that f is not identically equal to the identity map (otherwise the required property holds trivially), then $|f(x)-x|$ attains a positive maximum value in some point $t\in(0,1)$.
Let us consider the set $D=f'((0,1))$. By the Mean Value Theorem,

$$b:=\frac{f(t)}{t}=\frac{f(t)-f(0)}{t-0}\in D \quad\text{and}\quad a:=1-\frac{f(t)-t}{1-t}=\frac{1-t-(f(t)-t)}{1-t}=\frac{f(1)-f(t)}{1-t}\in D.$$

Note that $b>1$ and $a<1$ if $f(t)>t$, and $b<1$ and $a>1$ if $f(t)>t$. Hence by Darboux's Theorem, there exists $r>0$ such that $(1-r,1+r)\subset D$. Moreover, for any $y\in(1,1+r)$, we have that $1/y\in(1/(1+r),1)\subset(1-r,1)$.
Finally select $m:=\lfloor n/2\rfloor$ different numbers $y_1,y_2,\ldots,y_m\in(1,1+r)$ and let $y_{m+k}:=1/y_k$ for $k=1,\ldots,m$. Take also $y_n=1$ if n is odd. Then $y_1,y_2,\ldots,y_n$ are n different values in D and therefore there exist n different numbers $c_1,c_2,\ldots,c_n\in(0,1)$, such that $f'(c_k)=y_k$ for $k=1,\ldots,n$ and

$$\prod_{k=1}^{n} f'(c_k)=\prod_{k=1}^{m}\left(y_k\cdot\frac{1}{y_k}\right)=1.$$

$\square$