Problem 11747
(American Mathematical Monthly, Vol.121, January 2014)

Proposed by J. C. Lagarias (USA).

Determine all $n \in \mathbb{N}$ *such that* $\lfloor n/k \rfloor$ *divides* n *for* $1 \le k \le n$. *Similarly, determine all* $n \in \mathbb{N}$ *such that* $\lceil n/k \rceil$ *divides* n *for* $1 \le k \le n$.

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We will show that

i) $\lfloor n/k \rfloor$ divides n for $1 \le k \le n$ iff $n \in \{1, 2, 3, 4, 6, 8, 12, 24\}$,

ii) $\lceil n/k \rceil$ divides n for $1 \le k \le n$ iff $n \in \{1, 2, 3, 4, 6, 8, 12\}$.

We divide the proof in four steps.

1) If $\lfloor n/k \rfloor$ divides n for $1 \le k \le n$ then n is divisible by any integer $2 \le a \le \sqrt{n} - 1$.
In fact, given $2 \le a \le \sqrt{n} - 1$,

$$a = \lfloor n/k \rfloor \iff a \le \frac{n}{k} < a + 1, \iff k \in \left(\frac{n}{a+1}, \frac{n}{a}\right] \subset [1, n].$$

Since

$$\frac{n}{a} - \frac{n}{a+1} = \frac{n}{a(a+1)} \ge \frac{n}{\sqrt{n}(\sqrt{n}-1)} > 1$$

it follows that there exists an integer $k \in [1, n]$ such that $a = \lfloor n/k \rfloor$.

2) If $\lceil n/k \rceil$ divides n for $1 \le k \le n$ then n is divisible by any integer $2 \le a \le \sqrt{n} - 1$.
In fact, given $2 \le a \le \sqrt{n} - 1$,

$$a = \lceil n/k \rceil \iff a - 1 < \frac{n}{k} \le a, \iff k \in \left[\frac{n}{a}, \frac{n}{a-1}\right) \subset [1, n].$$

Since

$$\frac{n}{a-1} - \frac{n}{a} = \frac{n}{a(a-1)} \ge \frac{n}{(\sqrt{n}-1)(\sqrt{n}-2)} > 1$$

it follows that there exists an integer $k \in [1, n]$ such that $a = \lceil n/k \rceil$.

3) If n is divisible by any integer $2 \le a \le \sqrt{n} - 1$ then $n < 64$.
Let $a = \lfloor \sqrt{n} \rfloor - 1$, then $\gcd(a, a-1) = 1$, $a(a-1)$ divides n and $n = qa(a-1)$ for some integer $q > 1$ (because $a(a-1) < n$). Hence

$$(a+2)^2 \ge n = qa(a-1) \ge 2a(a-1) \iff a^2 - 6a - 4 \le 0 \iff (a-3)^2 \le 13,$$

and $a \le 6$, that is $\lfloor \sqrt{n} \rfloor = a + 1 \le 7$, or $n < 64$.

4) It is easy to verify that our statement holds for $n < 64$.

□