Problem 11744
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Flip a fair coin until the start of the r-th run. Let Y be the number of runs consisting of one head. Find the expected value and variance of Y.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $r \geq 1$ and $s \geq 0$, let $P_T(r,s)$ and $P_H(r,s)$ the probability that a sequence of r runs and s single heads starts respectively with T and with H. Hence $P(r,s) = P_T(r,s) + P_H(r,s)$ is the probability that a sequence of r runs has s single heads. Then it is easy to see that

$$P_T(1,s) = \begin{cases} 1 & \text{if } s = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad P_T(2,s) = \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$P_H(1,s) = \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad P_H(2,s) = \begin{cases} 1/2 & \text{if } s \in \{0,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$P_T(r,s) = \frac{1}{2}P_T(r,s) + \frac{1}{2}P_H(r-1,s)\,, \quad P_H(r,s) = \frac{1}{2}P_H(r-2,s-1) + \frac{1}{2}P_T(r-1,s),$$

which implies

$$P_T(r,s) = P_H(r-1,s)\,, \quad P_H(r,s) = \frac{1}{2}P_H(r-2,s-1) + \frac{1}{2}P_H(r-2,s).$$

Therefore, we have that

$$P(1,s) = \begin{cases} 1/2 & \text{if } s \in \{0,1\}, \\ 0 & \text{otherwise,} \end{cases} \qquad P(2,s) = \begin{cases} 1/4 & \text{if } s = 0, \\ 3/4 & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}.$$

and

$$P(r,s) = \frac{1}{2}P(r-2,s-1) + \frac{1}{2}P(r-2,s).$$

By solving the linear recurrence, we obtain that

$$P(r,s) = \begin{cases} \dfrac{1}{2^{k+1}}\dbinom{k+1}{s} & \text{if } r = 2k+1, \\[2ex] \dfrac{1}{2^{k+1}}\left(1 + \dfrac{2s}{k}\right)\dbinom{k}{s} & \text{if } r = 2k. \end{cases}$$

Since

$$\sum_{s\geq 0}\binom{N}{s}s = 2^{N-1}N, \quad \sum_{s\geq 0}\binom{N}{s}s^2 = 2^{N-2}(N^2+N), \quad \sum_{s\geq 0}\binom{N}{s}s^3 = 2^{N-3}(N^3+3N^2),$$

it follows that

$$E(Y) = \sum_{s\geq 0} sP(r,s) = \frac{r+1}{4}, \ \mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \sum_{s\geq 0} s^2P(r,s) - E(Y)^2 = \frac{r+1}{8} - \frac{3(1+(-1)^r)}{32}.$$

□