Problem 11743
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Proposed by Francois Capaces (France).

Let n be a positive integer, let x be a real number, and let $B_n(x)$ be the n-by-n matrix with $2x$ in all diagonal entries, 1 in all sub- and super-diagonal entries, and 0 in all other entries. Compute the inverse, when it exists, of $B_n(x)$ as a function of x.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $n \geq 1$,

$$B_n(x) := \begin{pmatrix} 2x & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2x \end{pmatrix}$$

We note that $\det(B_1) = 2x$, $\det(B_2) = 4x^2 - 1$ and it is easy to see that and for $n \geq 3$,

$$\det(B_n) = 2x \det(B_{n-1}) - \det(B_{n-2}).$$

It is well known that this recurrence is solved by the *Chebyshev Polynomial of the Second Kind*

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\left(\frac{k\pi}{n+1} \right) \right).$$

So if x different from $\cos\left(\frac{k\pi}{n+1}\right)$ for $k = 1, \ldots, n$, then $B_n(x)$ is invertible.
The inverse can be determined by evaluating the co-factors which are easy to find thanks to the special structure of the matrix. The inverse is a symmetric matrix and the entries for $i \leq j$ are

$$[B_n(x)^{-1}]_{i,j} = \frac{(-1)^{i+j} U_{i-1}(x) U_{n-j}(x)}{U_n(x)}.$$

For more formulas of such inverse see the paper:

G. Y. Hu and R. F. O'Connell, *Analytical inversion of symmetric tridiagonal matrices*,
J. Phys. A: Math. Gen. 29 (1996) 1511-1513. □