Problem 11739
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Proposed by Fred Adams, Anthony Bloch, and Jeffrey Lagarias (USA).

Let $B(x) = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$. *Consider the infinite matrix product*

$$M(t) = \prod_{n=1}^{\infty} B(p_n^{-t}),$$

where p_n *is the* n*th prime. Evaluate* $M(2)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $N \geq 0$, let

$$M_N(2) = \prod_{n=1}^{N} B(p_n^{-2}) = \begin{bmatrix} a(N) & b(N) \\ b(N) & a(N) \end{bmatrix}.$$

Then $a(0) = 1$, $b(0) = 0$ and

$$a(N) = a(N-1) + \frac{b(N-1)}{p_N^2} \quad \text{and} \quad b(N) = b(N-1) + \frac{a(N-1)}{p_N^2}.$$

Hence,

$$a(N) + b(N) = (a(N-1) + b(N-1)) \left(1 + \frac{1}{p_N^2}\right) = \prod_{1}^{N} \left(1 + \frac{1}{p_n^2}\right)$$
$$= \prod_{1}^{N} \left(1 - \frac{1}{p_n^4}\right) \Big/ \left(1 - \frac{1}{p_n^2}\right) \overset{N\to\infty}{\longrightarrow} \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2},$$

and

$$a(N) - b(N) = (a(N-1) - b(N-1)) \left(1 - \frac{1}{p_N^2}\right) = \prod_{1}^{N} \left(1 - \frac{1}{p_n^2}\right) \overset{N\to\infty}{\longrightarrow} \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Finally, $M(2) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ with

$$a = \lim_{N\to\infty} a(N) = \frac{1}{2}\left(\frac{15}{\pi^2} + \frac{6}{\pi^2}\right) = \frac{21}{2\pi^2}, \quad \text{and} \quad b = \lim_{N\to\infty} b(N) = \frac{1}{2}\left(\frac{15}{\pi^2} - \frac{6}{\pi^2}\right) = \frac{9}{2\pi^2}.$$

□