Problem 11736
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Proposed by Mircea Merca (Romania).

For $n \geq 1$, let f be the symmetric polynomial in variables $x_1, \ldots, x_n$, given by

$$f(x_1, \ldots, x_n) = \sum_{k=0}^{n-1} (-1)^{k+1} e_k(x_1 + x_1^2, x_2 + x_2^2, \ldots, x_n + x_n^2),$$

where e_k is the kth elementary polynomial in n variables. Also, let ω be a primitive nth root of unity. Prove that

$$f(1, \omega, \omega^2, \ldots, \omega^{n-1}) = L_n - L_0,$$

where L_k is the k-th Lucas number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that

$$P(z) = \prod_{k=1}^{n} (z - z_k) = \sum_{k=0}^{n} (-1)^k e_k(z_1, \ldots, z_n) z^{n-k}.$$

Morever $L_n = (\tau_+)^n + (\tau_-)^n$ with $\tau_\pm = (1 \pm \sqrt{5})/2$.
Hence, by letting $z_k = \omega^k + \omega^{2k}$, we obtain

$$f(1, \omega, \omega^2, \ldots, \omega^{n-1}) = P(0) - P(1) = ((-1)^n - 1) - (1 - L_n + (-1)^n) = L_n - 2 = L_n - L_0$$

because

$$P(0) = \prod_{k=0}^{n-1} (-(\omega^k + \omega^{2k})) = \omega^{n(n-1)/2} \prod_{k=0}^{n-1} (-1 - \omega^k) = \omega^{n(n-1)/2} ((-1)^n - 1) = (-1)^n - 1,$$

and

$$\begin{aligned} P(1) &= \prod_{k=0}^{n-1} (1 - (\omega^k + \omega^{2k})) = (-1)^n \prod_{k=0}^{n-1} (-\tau_+ - \omega^k)(-\tau_- - \omega^k) \\ &= (-1)^n \prod_{k=0}^{n-1} (-\tau_+ - \omega^k) \prod_{k=0}^{n-1} (-\tau_- - \omega^k) = (-1)^n ((-\tau_+)^n - 1)((-\tau_-)^n - 1) \\ &= (-1)^n ((-1)^n - (-1)^n ((\tau_+)^n + (\tau_-)^n) + 1) = 1 - L_n + (-1)^n. \end{aligned}$$

□