Problem 11732
(American Mathematical Monthly, Vol.120, November 2013)

Proposed by Marcel Chirita (Romania).

Let a and b be real, with $1 < a < b$, and let m and n be real with $m \neq 0$. Find all continuous functions f from $[1, \infty)$ to $\mathbb{R}$ such that for $x \geq 0$,

$$f(a^x) + f(b^x) = mx + n.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $g(t) := (f((ab)^t) - n/2)/m$ for $t \geq 0$. Then for $x \geq 0$

$$m(g(rx) + g((1-r)x)) + n = mg(rx) + \frac{n}{2} + mg((1-r)x) + \frac{n}{2} = f(a^x) + f(b^x)$$

where $r := \log_{ab} b > \log_{ab} a = 1 - r > 0$. Therefore, we have to solve for $x \geq 0$

$$g(rx) + g((1-r)x) = x.$$

Let $g(t) := t + h(t)$ then $h(rx) = -h((1-r)x)$. Hence, for $t \geq 0$, and for any positive integer n,

$$h(t) = -h(\alpha t) = h(\alpha^2 t) = \cdots = (-1)^n h(\alpha^n t)$$

where $\alpha := (1-r)/r \in (0,1)$. Note that $h(0) = (-1)^n h(0)$ implies that $h(0) = 0$.
Since f is continuous at 1 then h is continuous at 0 and for $t \geq 0$

$$h(t) = \lim_{n \to \infty} (-1)^n h(\alpha^n t) = 0.$$

because $\alpha^n t \to 0$ and $h(\alpha^n t) \to h(0) = 0$.
So the unique solution is $g(t) = t$ for $t \geq 0$, that is for $x \geq 1$,

$$f(x) = f((ab)^{\log_{ab}(x)}) = m \log_{ab}(x) + \frac{n}{2}.$$

Note that in this case the identity

$$f(a^x) + f(b^x) = mx + n$$

holds for any $x \in \mathbb{R}$. □