Problem 11730
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Let p be the partition function (counting the ways to write n as a sum of positive integers), extended so that $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Prove that

$$\sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right) = 1.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We prove by induction that $S(n) = 1$ for all non-negative integers n, where

$$S(n) = \sum_{k=0}^{\infty}\sum_{j=0}^{2k}(-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right).$$

Now, $S(0) = p(0) = 1$ and for $n \geq 0$

$$\begin{aligned}
S(n+1) - S(n) &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{2k}(-1)^k p\left(n + 1 - \frac{k(3k+1)}{2} - j\right) - \sum_{j=0}^{2k}(-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right)\right)\\
&= \sum_{k=0}^{\infty}\left(\sum_{j=-1}^{2k-1}(-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right) - \sum_{j=0}^{2k}(-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right)\right)\\
&= \sum_{k=0}^{\infty}\left((-1)^k p\left(n - \frac{k(3k+1)}{2} + 1\right) - (-1)^k p\left(n - \frac{k(3k+1)}{2} - 2k\right)\right)\\
&= \sum_{k=0}^{\infty}(-1)^k p\left(n + 1 - \frac{k(3k+1)}{2}\right) + \sum_{k=0}^{\infty}(-1)^{k+1} p\left(n - \frac{k(3k+1)}{2} - 2k\right)\\
&= \sum_{k=0}^{\infty}(-1)^k p\left(n + 1 - \frac{k(3k+1)}{2}\right) + \sum_{k=0}^{\infty}(-1)^{k+1} p\left(n - \frac{k(3k+5)}{2}\right)\\
&= \sum_{k=0}^{\infty}(-1)^k p\left(n + 1 - \frac{k(3k+1)}{2}\right) - \sum_{i=-\infty}^{-1}(-1)^{-i} p\left(n - \frac{(-i-1)(3(-i-1)+5)}{2}\right)\\
&= \sum_{k=0}^{\infty}(-1)^k p\left(n + 1 - \frac{k(3k+1)}{2}\right) - \sum_{i=-\infty}^{-1}(-1)^{i} p\left(n + 1 - \frac{i(3i+1)}{2}\right)\\
&= \sum_{k=-\infty}^{\infty}(-1)^k p\left(n + 1 - \frac{k(3k+1)}{2}\right) = [x^{n+1}]f(x) = 0,
\end{aligned}$$

where in the last step we used Euler's pentagonal number theorem

$$f(x) := \left(\sum_{n=-\infty}^{\infty} p(n)x^n\right)\cdot\left(\sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k+1)}{2}}\right) = 1.$$

□