Problem 11729
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Proposed by Vassilis Papanicolaou (Greece).

An integer n is called b-normal if all digits $0, 1, \ldots, b-1$ appear the same number of times in the base-b expansion of n. Let $\mathcal{N}_b$ be the set of all b-normal integers. Determine those b for which

$$\sum_{n \in \mathcal{N}_b}^{\infty} \frac{1}{n} < \infty.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\mathcal{N}_b(k)$ be the set of all b-normal integers such that all digits $0, 1, \ldots, b-1$ appear k times in their base-b expansions. Then

$$b^{bk-1} \leq \min(\mathcal{N}_b(k)) \leq \max(\mathcal{N}_b(k)) \leq b^{bk}, \quad \text{and} \quad |\mathcal{N}_b(k)| = \frac{(b-1)(bk)!}{b(k!)^b}.$$

Moreover $L \leq S \leq R$, where

$$S = \sum_{n \in \mathcal{N}_b}^{\infty} \frac{1}{n} = \sum_{k=1}^{\infty} \sum_{n \in \mathcal{N}_b(k)}^{\infty} \frac{1}{n}, \quad L = \sum_{k=1}^{\infty} \frac{|\mathcal{N}_b(k)|}{b^{bk}}, \quad R = \sum_{k=1}^{\infty} \frac{|\mathcal{N}_b(k)|}{b^{bk-1}} = bL,$$

which imply that S is convergent iff L is convergent.
By Stirling's approximation formula $n! \sim \sqrt{2\pi n}(n/e)^n$, there is a positive constant c_b such that

$$\frac{|\mathcal{N}_b(k)|}{b^{bk}} = \frac{(b-1)(bk)!}{b(k!)^b b^{bk}} \sim \frac{c_b}{k^{(b-1)/2}},$$

and we obtain that S is convergent iff $(b-1)/2 > 1$ that is $b > 3$. □