Problem 11728
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Proposed by Walter Blumberg (USA).

Let p be a prime congruent to 7 modulo 8. Prove that

$$\sum_{k=1}^{p}\left\lfloor\frac{k^2+k}{p}\right\rfloor=\frac{2p^2+3p+7}{6}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $r_n(m)$ be the remainder of the division of m by n then, since $m=n\lfloor m/n\rfloor+\mathrm{r}_n(m)$, it follows that

$$\sum_{k=1}^{p}\left\lfloor\frac{k^2+k}{p}\right\rfloor=\frac{1}{p}\sum_{k=1}^{p}(k^2+k)-\frac{1}{p}\sum_{k=1}^{p}r_p(k^2+k)=\frac{p^2+3p+2}{3}-\frac{1}{p}\sum_{k=1}^{p-1}r_p(k^2+k)$$

Note that $r_p(k^2+k)=j$ for some $j\in\{1,\ldots,p-1\}$ iff $k^2+k=j \pmod p$, that is iff $\Delta=1+4j$ is a square modulo p iff $\left(\frac{4j+1}{p}\right)=1$ where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Therefore the number of $k\in\{1,\ldots,p-1\}$ such that $r_p(k^2+k)=j$ is $\left(\left(\frac{4j+1}{p}\right)+1\right)$ (it gives 0 or 2) and

$$\sum_{k=1}^{p-1}r_p(k^2+k)=\sum_{j=1}^{p-1}j\left(\left(\frac{4j+1}{p}\right)+1\right)=\sum_{j=1}^{p-1}j\left(\frac{4j+1}{p}\right)+\frac{p^2-p}{2}.$$

Hence,

$$\sum_{k=1}^{p}\left\lfloor\frac{k^2+k}{p}\right\rfloor=\frac{p^2+3p+2}{3}-\frac{1}{p}\sum_{j=1}^{p-1}j\left(\frac{4j+1}{p}\right)-\frac{p-1}{2}=\frac{2p^2+3p+7}{6}-\frac{1}{p}\sum_{j=0}^{p-1}j\left(\frac{4j+1}{p}\right),$$

so suffices to show that

$$S_1=\sum_{j=0}^{p-1}j\left(\frac{4j+1}{p}\right)=0.$$

Let

$$A=\sum_{j=0}^{p-1}j\left(\frac{j}{p}\right),\qquad B=\sum_{j=0}^{p-1}\left(\frac{j}{p}\right),$$

$$U_i=\sum_{j=0}^{p-1}j\left(\frac{2j+i}{p}\right),V_i=\sum_{j=0}^{p-1}\left(\frac{2j+i}{p}\right)\text{ for }i=0,1,$$

$$S_i=\sum_{j=0}^{p-1}j\left(\frac{4j+i}{p}\right),T_i=\sum_{j=0}^{p-1}\left(\frac{4j+i}{p}\right)\text{ for }i=0,1,2,3.$$

Since $\sum_{j=0}^{p-1}\left(\frac{aj+b}{p}\right)=0$ when p be a prime not dividing a, it follows that $B=V_i=T_i=0$ because p does not divide 2 and 4. Moreover $p\equiv 7 \pmod 8$ yields

$$\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}=-1\quad\text{and}\quad\left(\frac{2}{p}\right)=(-1)^{\lfloor(p+1)/4\rfloor}=1.$$

Now, $U_0 = S_0 = A$ and

$$\sum_{r=0}^{2p-1} r\left(\frac{r}{p}\right) = \sum_{i=0}^{1}\sum_{j=0}^{p-1}(2j+i)\left(\frac{2j+i}{p}\right) = 2U_0 + 2U_1 + V_1 = 2A + 2U_1,$$

$$\sum_{r=0}^{2p-1} r\left(\frac{r}{p}\right) = \sum_{i=0}^{1}\sum_{r=0}^{p-1}(ip+r)\left(\frac{ip+r}{p}\right) = A + pB + A = 2A,$$

which imply that $S_2 = U_1 = 0$. Moreover

$$\sum_{r=0}^{4p-1} r\left(\frac{r}{p}\right) = \sum_{i=0}^{3}\sum_{j=0}^{p-1}(4j+i)\left(\frac{4j+i}{p}\right) = 4S_0 + 4S_1 + 4S_2 + 4S_3 + T_1 + T_2 + T_3 = 4A + 4S_1 + 4S_3,$$

$$\sum_{r=0}^{4p-1} r\left(\frac{r}{p}\right) = \sum_{i=0}^{3}\sum_{r=0}^{p-1}(ip+r)\left(\frac{ip+r}{p}\right) = A + pB + A + 2pB + A + 3pB + A = 4A,$$

which imply that $S_1 + S_3 = 0$. Finally

$$S_1 = \sum_{j=1}^{p-1} j\left(\frac{4j+1}{p}\right) = \sum_{j=1}^{p-1}(p-j)\left(\frac{4(p-j)+1}{p}\right) = -p\sum_{j=1}^{p-1}\left(\frac{4j-1}{p}\right) + \sum_{j=1}^{p-1} j\left(\frac{4j-1}{p}\right)$$

$$= -(p-1)\sum_{j=0}^{p-2}\left(\frac{4j+3}{p}\right) + \sum_{j=0}^{p-2} j\left(\frac{4j+3}{p}\right) = (p-1)\left(\frac{-1}{p}\right) - (p-1)T_3 - (p-1)\left(\frac{-1}{p}\right) + S_3 = S_3$$

and we obtain that $S_1 = S_3 = 0$. □