Problem 11725
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Let m be a positive integer. Show that, as $n \to \infty$,

$$\left|\ln(2) - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\right| = \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots + \frac{C_m}{n^m} + o\left(\frac{1}{n^m}\right),$$

where

$$C_k = (-1)^k \sum_{i=1}^{k} \frac{1}{2^i} \sum_{j=1}^{i} (-1)^j \binom{i-1}{j-1} j^{k-1}$$

for $1 \le k \le m$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By Euler-Maclaurin summation formula,

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \gamma + \frac{1}{2n} - \sum_{k=2}^{m} \frac{B_k}{k} \cdot \frac{1}{n^k} + o\left(\frac{1}{n^m}\right),$$

and it follows that

$$\begin{aligned}
\left|\ln(2) - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\right| &= \left|\ln(2) - \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n/2} \frac{1}{k}\right| \\
&= \left|\ln(2) - \ln(n) - \gamma - \frac{1}{2n} + \sum_{k=2}^{m} \frac{B_k}{k} \cdot \frac{1}{n^k}\right. \\
&\qquad \left. + \ln(n/2) + \gamma + \frac{1}{n} - \sum_{k=2}^{m} \frac{B_k}{k} \cdot \frac{2^k}{n^k} + o\left(\frac{1}{n^m}\right)\right| \\
&= \frac{1}{2n} - \sum_{k=2}^{m} \frac{(2^k - 1)B_k}{k} \cdot \frac{1}{n^k} + o\left(\frac{1}{n^m}\right).
\end{aligned}$$

Now $C_1 = 1/2$ and it suffices to show that $C_k = -(2^k - 1)B_k/k$ for $k \ge 2$.
The $(k-1)$-th *Euler polynomial* is given by

$$E_{k-1}(x) = \sum_{i=1}^{k} \frac{1}{2^{i-1}} \sum_{j=1}^{i} (-1)^{j-1} \binom{i-1}{j-1} (x + j - 1)^{k-1},$$

and it verifies the following known identities

$$E_{k-1}(x+1) + E_{k-1}(x) = 2x^{n-1} \quad \text{and} \quad E_{k-1}(x) = \frac{2}{k}\left(B_k(x) - 2^k B_k(x/2)\right)$$

where $B_k(x)$ is the k-th *Bernoulli polynomial*. Finally

$$C_k = \frac{(-1)^{k-1} E_{k-1}(1)}{2} = \frac{(-1)^k E_{k-1}(0)}{2} = \frac{(-1)^k \left(B_k(0) - 2^k B_k(0)\right)}{k} = -\frac{(2^k - 1)B_k}{k},$$

where the last equality holds because $B_k(0) = B_k$ and $B_k = 0$ if $k \ge 2$ and k is odd. □