Problem 11724
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Proposed by Andrew Cusumano (USA).

Let $f(n) = \sum_{k=1}^{n} k^k$ *and let* $g(n) = \sum_{k=1}^{n} f(k)$*. Find*

$$\lim_{n\to\infty}\left(\frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that

$$\lim_{n\to\infty}\left(\frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)}\right) = e.$$

Since $g(n) = \sum_{k=1}^{n}(n+1-k)k^k = \sum_{k=1}^{n} k(n+1-k)^{n+1-k}$, it follows that

$$\frac{g(n)}{n^n} = \sum_{k=1}^{n} \frac{k}{n^{k-1}}\left(1-\frac{k-1}{n}\right)^{n+1-k} = 1 + \frac{2}{n}\left(1-\frac{1}{n}\right)^{n-1} + h(n) = 1 + \frac{2e^{-1}}{n} + o\left(\frac{1}{n}\right).$$

because

$$0 \le h(n) = \sum_{k=3}^{n} \frac{k}{n^{k-1}}\left(1-\frac{k-1}{n}\right)^{n+1-k} \le \frac{3}{n^2} + \frac{4}{n^3} + \frac{1}{n^4}\sum_{k=5}^{n} k \le \frac{3}{n^2} + \frac{4}{n^3} + \frac{n^2+n}{2n^4}.$$

Hence

$$\begin{aligned}\frac{g(n+1)}{n^{n+1}} &= \left(1+\frac{1}{n}\right)^{n+1} \cdot \frac{g(n+1)}{(n+1)^{n+1}} = e\left(1+\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)\left(1+\frac{2e^{-1}}{n}+o\left(\frac{1}{n}\right)\right) \\ &= e + \frac{e+4}{2n} + o\left(\frac{1}{n}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{g(n+2)}{n^{n+2}} &= \left(1+\frac{2}{n}\right)^{n+2} \cdot \frac{g(n+2)}{(n+2)^{n+2}} = e^2\left(1+\frac{2}{n}+o\left(\frac{1}{n}\right)\right)\left(1+\frac{2e^{-1}}{n}+o\left(\frac{1}{n}\right)\right) \\ &= e^2 + \frac{2e^2+2e}{n} + o\left(\frac{1}{n}\right).\end{aligned}$$

Finally

$$\begin{aligned}\frac{g(n+2)}{g(n+1)} - \frac{g(n+1)}{g(n)} &= \frac{g(n+2)g(n) - g(n+1)^2}{g(n+1)g(n)} \\ &= \left(\frac{n}{e}+o(1)\right)\left[\left(e^2+\frac{2e^2+2e}{n}+o\left(\frac{1}{n}\right)\right)\left(1+\frac{2e^{-1}}{n}+o\left(\frac{1}{n}\right)\right) - \left(e+\frac{e+4}{2n}+o\left(\frac{1}{n}\right)\right)^2\right] \\ &= \left(\frac{n}{e}+o(1)\right)\left(\frac{e^2}{n}+o\left(\frac{1}{n}\right)\right) = e + o\left(\frac{1}{n}\right) \to e.\end{aligned}$$

□