Problem 11721
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Let p be a prime greater than 3, and let q be a complex number other than 1 such that $q^p = 1$. Evaluate

$$\sum_{k=1}^{p-1} \frac{(1-q^k)^5}{(1-q^{2k})^3(1-q^{3k})^2}.$$

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Let a, b, d be non-negative integers such that $\gcd(a,p) = 1$.
Let $j_0 \equiv -b/a \pmod p$ such that $j_0 \in \{0, 1, \ldots, p-1\}$. Since

$$\sum_{k=1}^{p-1} q^{k(aj+b)} = -1 + \begin{cases} p & \text{if } p \mid (aj+b) \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{q^{bk}}{(1-q^{ak}z)^d} &= = \sum_{k=1}^{p-1} q^{bk} \sum_{j\geq 0} \binom{j+d-1}{d-1} q^{akj} z^j \\
&= \sum_{j\geq 0} \binom{j+d-1}{d-1} z^j \sum_{k=1}^{p-1} q^{k(aj+b)} \\
&= p\sum_{s\geq 0} \binom{j_0+sp+d-1}{d-1} z^{j_0+sp} - \frac{1}{(1-z)^d} \\
&= \frac{p\sum_{s=0}^{d-1} c_s z^{j_0+sp}}{(1-z^p)^d} - \frac{1}{(1-z)^d}
\end{aligned}$$

where

$$\begin{aligned}
c_s &= [z^{j_0+sp}](1-z^p)^d \sum_{s_2\geq 0} \binom{j_0+s_2p+d-1}{d-1} z^{j_0+s_2p} \\
&= [z^{j_0+sp}] \sum_{s_1\geq 0} (-1)^{s_1} \binom{d}{s_1} z^{s_1 p} \sum_{s_2\geq 0} \binom{j_0+s_2p+d-1}{d-1} z^{j_0+s_2p} \\
&= \sum_{s_1+s_2=s} (-1)^{s_1} \binom{d}{s_1} \binom{j_0+s_2p+d-1}{d-1} \\
&= \sum_{k=0}^{s} (-1)^{s-k} \binom{d}{s-k} \binom{j_0+kp+d-1}{d-1}.
\end{aligned}$$

Let $z = 1 + w$, then

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{q^{bk}}{(1-q^{ak})^d} &= \lim_{w\to 0} \frac{p\sum_{s=0}^{d-1} c_s(1+w)^{j_0+sp} - \left(\frac{1-(1+w)^p}{-w}\right)^d}{(1-(1+w)^p)^d} \\
&= \lim_{w\to 0} \frac{p\sum_{s=0}^{d-1} c_s \binom{j_0+sp}{d} w^d + o(w^d) - (-1)^d \sum_{s=0}^{d} (-1)^s \binom{d}{s}\binom{sp}{2d} w^d + o(w^d)}{(-pw+o(w))^d} \\
&= \frac{1}{p^d}\left((-1)^d p \sum_{s=0}^{d-1} c_s \binom{j_0+sp}{d} - \sum_{s=0}^{d} (-1)^s \binom{d}{s}\binom{sp}{2d}\right).
\end{aligned}$$

Hence

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{(1-q^{2k})^3} = \begin{cases} -(p-1)(p-3)/8 & \text{if } b=0 \\ (p-1)(p+1)/24 & \text{if } b=1,4 \\ -(p-1)(p+1)/24 & \text{if } b=2,5 \\ 0 & \text{if } b=3 \end{cases},$$

and

$$\sum_{k=1}^{p-1} \frac{q^{bk}}{(1-q^{3k})^2} = \begin{cases} -(p-1)(p-5)/12 & \text{if } b=0 \\ (p+5)(p-1)/36 & \text{if } b=1,5 \text{ and } p\equiv 1 \pmod 3 \\ (p-5)(p+1)/36 & \text{if } b=1,5 \text{ and } p\equiv -1 \pmod 3 \\ (p-1)^2/36 & \text{if } b=2,4 \text{ and } p\equiv 1 \pmod 3 \\ (p+1)^2/36 & \text{if } b=2,4 \text{ and } p\equiv -1 \pmod 3 \\ -(p-1)(p+1)/12 & \text{if } b=3 \end{cases}.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(1-q^k)^5}{(1-q^{2k})^3(1-q^{3k})^2} &= \sum_{k=1}^{p-1} \frac{4-8q^k+q^{2k}+5q^{3k}-q^{4k}-q^{5k}}{(1-q^{2k})^3} \\ &\quad + \sum_{k=1}^{p-1} \frac{-3+3q^k+2q^{3k}-q^{4k}-q^{5k}}{(1-q^{3k})^2} \\ &= \begin{cases} -(55p-1)(p-1)/72 & \text{if } p\equiv 1 \pmod 3 \\ -(11p-1)(5p-1)/72 & \text{if } p\equiv -1 \pmod 3 \end{cases}. \end{aligned}$$

Note that the result is always a negative integer. □