Problem 11720
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Let $E_n(t)$ be the Eulerian polynomial defined by

$$\sum_{k=0}^{\infty}(k+1)^n t^k = \frac{E_n(t)}{(1-t)^{n+1}},$$

and let B_n be the nth Bernoulli number. Show that

$$(E_{n+1}(t)-(1-t)^n)B_n$$

is a polynomial with integer coefficients.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that for $n \geq 1$,

$$E_n(t) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle t^k$$

where

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k}(-1)^j \binom{n+1}{j}(k+1-j)^n$$

is the so-called *Eulerian number* which gives the number of permutations of $\{1,2,\ldots,n\}$ having k permutation ascents.
The nth Bernoulli number B_n is a rational number. If n is odd then $B_1 = -1/2$ and $B_n = 0$ for $n > 1$. If n is even then $B_0 = 1$ and for $n > 0$ the denominator is given by the product of the primes p such that $p-1$ divides n.
Therefore for $n \in \{0,1\}$,

$$(E_1(t)-1)B_0 = 0, \quad \text{and} \quad (E_2(t)-(1-t))B_1 = -t.$$

Moreover, the polynomial $(E_{n+1}(t)-(1-t)^n)B_n \in \mathbb{Z}[x]$ for all $n \geq 2$ iff for any prime p such that $p-1$ divides $n = 2m$

$$\left\langle {2m+1 \atop k} \right\rangle - (-1)^k \binom{2m}{k} \equiv 0 \pmod{p}.$$

Indeed, by Fermat's little theorem, for any integer x, $x^{2m+1} = x^{q(p-1)+1} \equiv x \pmod{p}$ which implies

$$\begin{aligned}
\left\langle {2m+1 \atop k} \right\rangle &= \sum_{j=0}^{k}(-1)^j \binom{2m+2}{j}(k+1-j)^{2m+1} \\
&\equiv \sum_{j=0}^{k}(-1)^j \binom{2m+2}{j}(k+1-j) \pmod{p} \\
&= (k+1)\sum_{j=0}^{k}(-1)^j \binom{2m+2}{j} + (2m+2)\sum_{j=1}^{k}(-1)^{j-1}\binom{2m+1}{j-1} \\
&= (k+1)(-1)^k\binom{2m+1}{k} + (2m+2)(-1)^{k-1}\binom{2m}{k-1} = (-1)^k\binom{2m}{k}.
\end{aligned}$$

□