Problem 11716
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Proposed by Oliver Knill (USA).

Let $\alpha = (\sqrt{5}-1)/2$. Let p_n and q_n be the numerator and denominator of the nth continued fraction convergent to α. Thus, $p_n = F_{n-1}$ and $q_n = F_n$, where F_n is the nth Fibonacci number defined by the recurrence $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. Show that

$$\sqrt{5}\left(\alpha - \frac{p_n}{q_n}\right) = \sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{q_n^{2k+2} 5^k},$$

where C_k denotes the kth Catalan number, given by $C_k = \frac{1}{k+1}\binom{2k}{k}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since the generating function for the Catalan numbers is

$$f(z) := \sum_{k=0}^{\infty} C_k z^k = \frac{1-\sqrt{1-4z}}{2z} \qquad \text{for } |z| < 1/4,$$

it follows that

$$\sum_{k=0}^{\infty} \frac{(-1)^{(n+1)(k+1)} C_k}{q_n^{2k+2} 5^k} = \frac{(-1)^{n+1}}{q_n^2} f\left(\frac{(-1)^{n+1}}{q_n^2 5}\right) = \frac{5q_n - \sqrt{25q_n^2 + 20(-1)^n}}{2q_n}$$

because $|(-1)^{n+1}/(q_n^2 5)| \leq 1/5$. Hence, we have to verify that

$$\sqrt{5}\left((\sqrt{5}-1)q_n - 2p_n\right) = 5q_n - \sqrt{25q_n^2 + 20(-1)^n},$$

that is

$$q_n + 2p_n = \sqrt{5q_n^2 + 4(-1)^n}.$$

Now by letting $p_n = F_{n-1}$ and $q_n = F_n$, the above equation is equivalent to

$$F_{n-1}^2 - F_n^2 + F_{n-1}F_n = F_{n-1}F_n - F_{n-2}F_{n+1} = (-1)^n,$$

which is a well known identity. $\square$