Problem 11715
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Proposed by M. Stofka (Slovakia).

Prove that

$$\sum_{k=0}^{\infty}\frac{1}{(6k+1)^5}=\frac{1}{2}\left(\frac{2^5-1}{2^5}\cdot\frac{3^5-1}{3^5}\,\zeta(5)+\frac{11\pi^5}{8\cdot 3^5\sqrt{3}}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $1\leq j\leq 5$, let

$$S_j=\sum_{k=0}^{\infty}\frac{1}{(6k+j)^5}.$$

Then it is easy to see that

$$S_3=\frac{1}{3^5}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^5}=\frac{1}{3^5}\left(\zeta(5)-\sum_{k=1}^{\infty}\frac{1}{(2k)^5}\right)=\frac{2^5-1}{6^5}\,\zeta(5),$$

$$S_2+S_4=\frac{1}{2^5}\left(\sum_{k=0}^{\infty}\frac{1}{(3k+1)^5}+\sum_{k=0}^{\infty}\frac{1}{(3k+2)^5}\right)=\frac{1}{2^5}\left(\zeta(5)-\sum_{k=1}^{\infty}\frac{1}{(3k)^5}\right)=\frac{3^5-1}{6^5}\,\zeta(5),$$

$$S_1+S_2+S_3+S_4+S_5=\zeta(5)-\sum_{k=1}^{\infty}\frac{1}{(6k)^5}=\frac{6^5-1}{6^5}\,\zeta(5).$$

Hence

$$S_1+S_5=\frac{6^5-3^5-2^5+1}{6^5}\,\zeta(5)=\frac{2^5-1}{2^5}\cdot\frac{3^5-1}{3^5}\,\zeta(5).$$

By the the partial fraction expansion of the cotangent function, we know that for $z\in\mathbb{C}\setminus\mathbb{Z}$,

$$\frac{\pi}{\tan(\pi z)}=\sum_{k=-\infty}^{\infty}\frac{1}{z-k}.$$

Moreover, the following power series converges for all z such that $|z|<1$, and since the sum of the terms with $n>1$ converges absolutely, we may interchange the order of summation. So we have

$$F(z):=\sum_{n=1}^{\infty}\left(\sum_{k=-\infty}^{\infty}\frac{1}{(6k+1)^n}\right)z^n=\sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{z^n}{(6k+1)^n}=-z\sum_{k=-\infty}^{\infty}\frac{1}{z-(6k+1)}=-\frac{\pi z}{6\tan(\pi(z-1)/6)},$$

which implies that

$$S_1-S_5=\sum_{k=-\infty}^{\infty}\frac{1}{(6k+1)^5}=\frac{1}{5!}\frac{d^5F}{dz^5}(0)=\frac{11\pi^5}{8\cdot 3^5\sqrt{3}}.$$

Finally, we obtain

$$S_1=\frac{(S_1+S_5)+(S_1-S_5)}{2}=\frac{1}{2}\left(\frac{2^5-1}{2^5}\cdot\frac{3^5-1}{3^5}\,\zeta(5)+\frac{11\pi^5}{8\cdot 3^5\sqrt{3}}\right),$$

$$S_5=\frac{(S_1+S_5)-(S_1-S_5)}{2}=\frac{1}{2}\left(\frac{2^5-1}{2^5}\cdot\frac{3^5-1}{3^5}\,\zeta(5)-\frac{11\pi^5}{8\cdot 3^5\sqrt{3}}\right).$$

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