Problem 11713
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Proposed by Mihaly Bencze (Romania).

Let $x_1, \dots, x_n$ be nonnegative real numbers. Prove that

$$\prod_{k=1}^{n}(1+x_k) \leq 1+\sum_{k=1}^{n}\frac{1}{k!}\left(1-\frac{k}{2n}\right)^{k-1}\left(\sum_{k=1}^{n}x_k\right)^{k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let σ_k be k-th elementary symmetric polynomial of $x_1, \dots, x_n$,

$$\sigma_k = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k}.$$

By the Symmetric Mean inequality, it follows that

$$\left(\frac{\sigma_k}{\binom{n}{k}}\right)^{1/k} \leq \frac{\sigma_1}{n}.$$

Hence

$$\begin{aligned}\sigma_k \leq \binom{n}{k}\cdot\frac{\sigma_1^k}{n^k} &= \frac{(n-1)\cdots(n-(k-1))}{n^{k-1}}\cdot\frac{\sigma_1^k}{k!}\\ &\leq \frac{(((n-1)+\cdots+(n-(k-1)))/(k-1))^{k-1}}{n^{k-1}}\cdot\frac{\sigma_1^k}{k!} = \left(1-\frac{k}{2n}\right)^{k-1}\cdot\frac{\sigma_1^k}{k!}\end{aligned}$$

where we used the AM-GM inequality. Finally,

$$\prod_{k=1}^{n}(1+x_k) = 1+\sum_{k=1}^{n}\sigma_k \leq 1+\sum_{k=1}^{n}\left(1-\frac{k}{2n}\right)^{k-1}\cdot\frac{\sigma_1^k}{k!}.$$

□