Problem 11712
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In the game of Bulgarian solitaire, n identical coins are distributed into two piles, and a move takes one coin from each existing pile to form a new pile. Beginning with a single pile of size n, how many moves are needed to reach a position on a cycle (a position that will eventually repeat)? For example, $5 \to 41 \to 32 \to 221 \to 311 \to 32$, so the answer is 2 when $n = 5$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $n \geq 1$, let $M(n)$ be the required number of moves.
We will show that for $n = 1 + 2 + \cdots + (k-1) + r$ with $1 \leq r \leq k$, then

$$M(n) = n - k = n - \left\lfloor \frac{\sqrt{8n-7}+1}{2} \right\rfloor .$$

The first 20 terms of the sequence are: 0, 0, 1, 1, 2, 3, 3, 4, 5, 6, 6, 7, 8, 9, 10, 10, 11, 12, 13, 14.
We represent the coins in the piles as a partition of n, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 1$ and $\sum_{i=1}^{s} \lambda_i = n$. For such a partition λ we associate a $(0,1)$-array where the columns correspond to the parts of λ,

$$A_\lambda = [a_{i,j}] \quad \text{where} \quad a_{i,j} = \begin{cases} 1 & \text{if } j \leq s \text{ and } i \leq \lambda_j, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{B}(\lambda)$ the partition obtained from λ after one move. Then, according to the references, $A_{\mathcal{B}(\lambda)}$ can be obtained from A_λ by a *shifting process*: first we shift the entries of each diagonal of weight $d \geq 1$, $\{a_{i,d+1-i}\}_{i \in [1,d]}$, from left to right, then we remove all zero entries in the first column and shift the corresponding rows from right to left.

$$A_{(4,2)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{(3,2,1)}$$

Moreover, it has been shown that a partition λ of n is cyclic under the action of $\mathcal{B}$ if and only if it has the form

$$(k-1+\delta_{k-1}, k-2+\delta_{k-2}, \dots, 2+\delta_2, 1+\delta_1, \delta_0),$$

where each δ_i is 0 or 1 and $\sum_{i=0}^{k-1} \delta_i = r$. Let

$$f(A_\lambda) = \sum_{i+j \leq k+1} a_{i,j},$$

then, by the definition of the shifting process and the characterization of cyclic partitions, it is easy to verify that for $j \geq 0$,

$$f(A_{\mathcal{B}^{j+1}(n)}) = f(A_{\mathcal{B}^{j}(n)}) + \begin{cases} 0 & \text{if } \mathcal{B}^j(n) \text{ is cyclic,} \\ 1 & \text{otherwise.} \end{cases}$$

Note that this property could not hold under the weaker assumption that the starting partion is a generic partition of n (instead of (n)). Finally, since

$$f(A_{(n)}) = k \quad \text{and} \quad f(A_{\mathcal{B}^{M(n)}(n)}) = n,$$

it follows that the first cyclic partition is attained after $n - k$ steps. $\square$

References

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