**Problem 11711**
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Proposed by J. A. Grzesik (USA).

*Show, for integers $n$ and $k$ with $n \geq 2$ and $1 \leq k \leq n$, that*

$$(-1)^{n-k}\binom{n}{k}k\sum_{j=1,j\neq k}^{n}\frac{1}{k-j} = -\sum_{j=1,j\neq k}^{n}(-1)^{n-j}\binom{n}{j}\frac{j}{k-j}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By partial fraction decomposition, we have that for any $z \notin \{1,2,\dots,n\}\setminus\{k\}$

$$n!\prod_{j=1,j\neq k}^{n}\frac{1}{(z-j)} = \sum_{j=1,j\neq k}^{n}(-1)^{n-j}\binom{n}{j}\frac{j(j-k)}{z-j}.$$

Then we differentiate with respect to $z$ both sides,

$$n!\prod_{j=1,j\neq k}^{n}\frac{1}{(z-j)}\sum_{j=1,j\neq k}^{n}\frac{-1}{z-j} = \sum_{j=1,j\neq k}^{n}(-1)^{n-j}\binom{n}{j}\frac{j(k-j)}{(z-j)^2}.$$

Finally, by letting $z=k$, we obtain

$$(-1)^{n-k}\binom{n}{k}k\sum_{j=1,j\neq k}^{n}\frac{1}{k-j} = -\sum_{j=1,j\neq k}^{n}(-1)^{n-j}\binom{n}{j}\frac{j}{k-j}.$$

□