Problem 11709
(American Mathematical Monthly, Vol.120, May 2013)

Proposed by Moubinool Omarjee (France).

Find

$$I=\int_{x=0}^{+\infty}\frac{1}{x}\int_{y=0}^{x}\frac{\cos(x-y)-\cos(x)}{y}\,dy\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\begin{aligned}
I&=\int_{x=0}^{+\infty}\frac{\cos(x)}{x}\left(\int_{y=0}^{x}\frac{\cos(y)-1}{y}\right)dy\,dx+\int_{x=0}^{+\infty}\frac{\sin(x)}{x}\left(\int_{y=0}^{x}\frac{\sin(y)}{y}\,dy\right)dx\\
&=\int_{y=0}^{+\infty}\frac{1-\cos(y)}{y}\,\mathrm{Ci}(y)\,dy+\int_{x=0}^{+\infty}\frac{\sin(x)}{x}\,\mathrm{Si}(x)\,dx
\end{aligned}$$

where

$$\mathrm{Ci}(y)=-\int_{x=y}^{+\infty}\frac{\cos(x)}{x}\,dx\quad\text{and}\quad\mathrm{Si}(x)=\int_{y=0}^{x}\frac{\sin(y)}{y}\,dy.$$

Now

$$\int_{x=0}^{+\infty}\frac{\sin(x)}{x}\,\mathrm{Si}(x)\,dx=\frac{1}{2}\left[\mathrm{Si}(x)^2\right]_{x=0}^{+\infty}=\frac{1}{2}\left(\int_{x=0}^{+\infty}\frac{\sin(x)}{x}\,dx\right)^2=\frac{\pi^2}{8}.$$

Moreover, for $t>0$, by letting

$$F(t)=\int_{y=0}^{+\infty}\frac{1-\cos(y)}{y}\,\mathrm{Ci}(ty)\,dy,$$

we obtain

$$\int_{y=0}^{+\infty}\frac{1-\cos(y)}{y}\,\mathrm{Ci}(ty)\,dy=F(1)=\int_{+\infty}^{1}F'(t)\,dt=\int_{+\infty}^{1}\ln\left(1-\frac{1}{t^2}\right)\frac{dt}{2t}=\frac{1}{4}\int_0^1\frac{\ln(1-s)}{s}\,ds=\frac{\pi^2}{24}$$

because

$$\begin{aligned}
F'(t)&=\int_{y=0}^{+\infty}\frac{1-\cos(y)}{y}\cdot\frac{\cos(ty)}{t}\,dy\\
&=\frac{1}{2t}\int_{y=0}^{+\infty}\frac{2\cos(ty)-\cos((t+1)y)-\cos((t-1)y)}{y}\,dy\\
&=\frac{1}{2t}\int_{y=0}^{+\infty}\frac{\cos(ty)-\cos((t+1)y)}{y}\,dy+\frac{1}{2t}\int_{y=0}^{+\infty}\frac{\cos(ty)-\cos((t-1)y)}{y}\,dy\\
&=\frac{1}{2t}\ln\left(\frac{t+1}{t}\right)+\frac{1}{2t}\ln\left(\frac{t-1}{t}\right)=\frac{1}{2t}\ln\left(1-\frac{1}{t^2}\right).
\end{aligned}$$

Hence

$$I=\frac{\pi^2}{24}+\frac{\pi^2}{8}=\frac{\pi^2}{6}.$$

□