Problem 11707
(American Mathematical Monthly, Vol.120, May 2013)

Proposed by José Luis Palacios (Venezuela).

For $N \geq 1$, consider the following random walk on the $(N+1)$-cycle with vertices labeled $0, 1, \ldots, N$. The walk begins at vertex 0 and continues until every vertex has been visited and the walk returns to vertex 0. Prove that the expected number of visits to any vertex other than 0 is $(2N+1)/3$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We divide our proof in several claims.

1) The expected number of steps needed to visit all vertices is $E = N(N+1)/2$.

   Consider n points arranged in a line and assume that the random walk starts from vertex j and let z_j the expected number of steps needed to visit for the first time vertex 1 or vertex n. Then the following equations holds:

$$z_1 = 0, z_n = 0,\ z_j = (z_{j+1} + z_{j-1})/2 + 1 \text{ for } 1 < j < n$$

   which yield $z_j = j(n+1) - j^2 - n$. Hence, going back to our the $(N+1)$-cycle, after $n-2$ verteces have been visited, a new one will be visited after $z_2 = z_{n-1} = n-2$. Indeed the already visited verteces are contiguous, the starting point is vertex 2 or vertex $n-1$, and the new one will be vertex 1 or vertex n. So the expected total number of steps needed to visit all vertices is

$$E = 1 + 2 + \cdots + N = N(N+1)/2.$$

2) The probability that the last visited vertex on the $(N+1)$-cycle is $j \in \{1, 2, \ldots, N\}$ is given by $p_i = 1/N$.

   Cut the circle in j and duplicate vertex j. We obtain $N+2$ points arranged in a line

$$j_L, j+1, \ldots, N, 0, 1, 2, \ldots, j-1, j_R$$

   Now j will be the last visited vertex if one of these disjoint events happens

   i) starting from 0, the walk visits $j-1$ before visiting $j+1$, then, starting from $j-1$, it visits $j+1$ before visiting j_R;

   ii) starting from 0, the walk visits $j+1$ before visiting $j-1$, then, starting from $j+1$, it visits $j-1$ before visiting j_L.

   Now it is known that if we have n points arranged in a line and a random walk starts form vertex j then vertex 1 will be visited before vertex n with probability $(n-j)/(n-1)$. Therefore

$$p_j = \frac{N-j}{N-1} \cdot \frac{1}{N} + \frac{j-1}{N-1} \cdot \frac{1}{N} = \frac{1}{N}.$$

3) The expected number of steps needed to go back from vertex j to 0 is $E_j = j(N+1) - j^2$.

   The following equations holds:

$$E_0 = 0, E_N = (E_0 + E_{N-1})/2 + 1,\ E_j = (E_{j+1} + E_{j-1})/2 + 1 \text{ for } 0 < j < N$$

   which yield $E_j = j(N+1) - j^2$.

Hence the expected number of steps needed to visit all vertices and then to go back to vertex 0 is

$$E + \sum_{j=1}^{N} p_j E_j = \frac{N(N+1)}{2} + \frac{1}{N} \sum_{j=1}^{N} (j(N+1) - j^2) = \frac{(N+1)(2N+1)}{3}.$$

Since the expected number of visits to any vertex is uniformly distributed we obtain that it is equal to the previous number divided by the number of vertices, that is $(2N+1)/3$. $\square$