Problem 11705
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Proposed by John Loase (USA).

Let $C(n)$ be the number of distinct multisets of two or more primes that sum to n. Prove that $C(n+1) \geq C(n)$ for all n.

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The first terms of the sequence $\{C_n\}_{n\geq 1}$ are:

$$0, 0, 0, 1, 1, 2, 2, 3, 4, 5, 5, 7, 8, 10, 12, 14, 16, 19, 22.$$

The inequality holds trivially for all $1 \leq n \leq 7$. Now we assume $n \geq 8$ and we show that any prime partition P of n generates a different prime partition P' of $n+1$.

1) If P is made of 2s then n is even. Let q and r be respectively the quotient and the remainder of $n/2$ divided by 3. Note that $q \geq 2$.

   1.1) If $r = 0$ we generate P' by replacing $q-1$ times $2, 2, 2$ with $3, 3$ and one time $2, 2, 2$ with 7.

   1.2) If $r = 1$ we generate P' by replacing q times $2, 2, 2$ with $3, 3$ and one time 2 with 3.

   1.3) If $r = 2$ we generate P' by replacing q times $2, 2, 2$ with $3, 3$ and one time $2, 2$ with 5.

2) If P contains at least an odd prime then let p be the smallest one. We generate P' by replacing p with $(p+1)/2$ copies of 2.

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