Problem 11704
(American Mathematical Monthly, Vol.120, April 2013)

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Let S_{2n} denote the symmetric group of all permutations of $\{1,\dots,2n\}$ and let T_{2n} denote the set of all fixed-point-free involutions in S_{2n}. Choose u and v from T_{2n} at random (any element of T_{2n} being as likely as any other) and independently. What is the probability that 1 and 2 will be in the same cycle of the permutation uv?

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The numbers 1 and 2 will be in the same cycle of the permutation uv if and only if there exist $1 \leq k \leq n-1$ and $a_1,\dots,a_{2k-1}$ distinct numbers in $\{3,\dots,2n\}$ such that, u has 2-cycles

$$(1,a_1),(a_2,a_3),\dots,(a_{2k-2},a_{2k-1})$$

and v has 2-cycles

$$(a_1,a_2),(a_3,a_4),\dots,(a_{2k-1},2).$$

These elements can be chosen in $(2n-2)(2n-3)\cdots(2n-2k)$ ways. Moreover, the fixed-point-free permutations u and v can be both completed in $(2n-2k-1)!!$ ways.
Hence, for $n \geq 2$, the probability that 1 and 2 will be in the same cycle of the permutation uv is equal to

$$\begin{aligned} p_n &= \frac{1}{((2n-1)!!)^2}\sum_{k=1}^{n-1}(2n-2)(2n-3)\cdots(2n-2k)((2n-2k-1)!!)^2 \\ &= \sum_{k=1}^{n-1}\frac{(2n-2)(2n-3)\cdots(2n-2k)}{(2n-1)^2(2n-3)^2\cdots(2n-2k+1)^2} = \frac{2n-2}{(2n-1)^2}\left(1+(2n-3)p_{n-1}\right) \end{aligned}$$

Since $p_1 = 0$, it is easy to verify that the above recurrence is satisfied by

$$p_n = \frac{2(n-1)}{3(2n-1)}.$$

The first values of p_n are: $0, \frac{2}{9}, \frac{4}{15}, \frac{2}{7}, \frac{8}{27}, \frac{10}{33}$. □