Problem 11700
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Let n be an integer greater than 1. Let a, b, and c be complex numbers with

$$a + b + c = a^n + b^n + c^n = 0$$

Prove that the absolute values of a, b, and c cannot be distinct.

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We may assume that at least one of the given complex numbers is different from zero, say a.
Let $z = b/a$ then $c/a = -(1+z)$. Let

$$P(z) := 1 + z^n + (-1)^n(z+1)^n.$$

It suffices to prove that if $z \in \mathcal{Z} := \{z : P(z) = 0\}$ then $z \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ where

$\mathcal{S}_1 := \{z : |z| = 1\}$ $(|a| = |b|)$, or $\mathcal{S}_2 := \{z : |z+1| = 1\}$ $(|a| = |c|)$, or $\mathcal{S}_2 := \{z : |z+1| = |z|\}$ $(|b| = |c|)$.

Let $G := \{z, 1/z, -(1+z), -1/(1+z), -z/(1+z), -(1+z)/z\}$ then it is easy to verify that $\mathcal{Z}$ and $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ are G-invariant. Now we locate all the zeros of P. Notice that

$$\{-2, -1, -1/2, 0, 1, e^{\pm 2\pi i/3}\} \cup \bigcup_{g \in G} g(\{e^{2\pi it} : t \in (1/3, 1/2)\}) \subset \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3.$$

Moreover

i) $1, -2, -1/2 \notin \mathcal{Z}$ for $n > 1$ and $0, -1 \in \mathcal{Z}$ iff n is odd;

ii) $e^{\pm 2\pi i/3} \in \mathcal{Z}$ with algebraic multiplicity 2 or 1 iff $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$ respectively;

iii) thanks to the G-invariance of $\mathcal{Z}$, if there are at least m roots of P in $\{e^{2\pi it} : t \in (1/3, 1/2)\}$ then there are at least $6m$ more roots of P in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

If we show that $m = \lfloor n/6 \rfloor - 1$ if $n \equiv 1 \pmod 6$ and $m = \lfloor n/6 \rfloor$ otherwise, we are done because for all $n > 1$ we found that the number of roots in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ is at least n if n is even and it is at least $n-1$ otherwise, which is precisely the degree of P. So there are no other roots.

$n \pmod 6$	0	1	2	3	4	5
i)	0	2	0	2	0	2
ii)	0	4	2	0	4	2
iii)	n	$n-7$	$n-2$	$n-3$	$n-4$	$n-5$
	n	$n-1$	n	$n-1$	n	$n-1$

The equation $P(e^{2\pi it}) = 0$ for $t \in (1/3, 1/2)$ is equivalent to

$$1 + e^{2\pi int} = (-1)^{n-1}(e^{2\pi int} + 1)^n$$

that is

$$\cos(\pi nt) = (-2)^{n-1}\cos(\pi t).$$

If $t \in (1/3, 1/2)$ then $|(-2)^{n-1}\cos(\pi t)| < 1/2$, whereas $|\cos(\pi nt)| \le 1$. Moreover, $\cos(\pi nt) = 1$ if $t = 2k/n$ and $\cos(\pi nt) = -1$ if $t = (2k+1)/n$. Hence, by the Intermediate Value Theorem, we will have at least m roots where $m+1$ is the number of integers in $[n/3, n/2]$, that is

$$m = \lfloor n/2 \rfloor - \lceil n/3 \rceil = \lfloor n/6 \rfloor - \begin{cases} 1 & \text{if } n \equiv 1 \pmod 6 \\ 0 & \text{otherwise.} \end{cases}.$$

□