Problem 11699
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Proposed by Bakir Farhi (Algeria).

Let $\{a_k\}_{k\geq 1}$ *be a strictly increasing sequence of positive integers such that* $\sum_{k=2}^{\infty}\frac{1}{a_k \ln a_k}$ *diverges. Prove that* $\mathrm{lcm}(a_1,\ldots,a_k)=\mathrm{lcm}(a_1,\ldots,a_{k+1})$ *for infinitely many* $k\geq 2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $m_k := \mathrm{lcm}(a_1,\ldots,a_k)$ and assume by contradiction that $m_k = m_{k+1}$ only for a finitely many $k\geq 2$. Hence, for some k_0, $m_{k+1} > m_k$ for any $k\geq k_0$. Since m_k divides m_{k+1}, it follows that for any $k\geq k_0$ there exist a prime q_k and a positive integer α_k such that $q_k^{\alpha_k}$ divides a_{k+1} and it does not divide any of the numbers $a_1, a_2, \ldots, a_k$. Note that if $k_0\leq k<j$ and $q_k = q_j$ then $\alpha_k<\alpha_j$. This means that, if $p_1, p_2, \ldots$ is the sequence of all primes in increasing order then

$$\begin{aligned}\sum_{k=k_0}^{\infty}\frac{1}{a_{k+1}\ln a_{k+1}} &\leq \sum_{k=k_0}^{\infty}\frac{1}{q_k^{\alpha_k}\ln(q_k^{\alpha_k})}\\ &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{p_k^j\ln(p_k^j)}\\ &\leq \sum_{k=1}^{\infty}\frac{1}{\ln p_k}\sum_{j=1}^{\infty}\frac{1}{jp_k^j}\\ &\leq \sum_{k=1}^{\infty}\frac{-\ln(1-1/p_k)}{\ln p_k}.\end{aligned}$$

Now, by the Prime Number Theorem, $p_k\sim k\ln k$ and

$$\frac{-\ln(1-1/p_k)}{\ln p_k}\sim\frac{1}{p_k\ln p_k}\sim\frac{1}{k(\ln k)^2}.$$

Hence the series with non negative terms

$$\sum_{k=1}^{\infty}\frac{-\ln(1-1/p_k)}{\ln p_k}$$

is convergent which contradicts the hypothesis that $\sum_{k=2}^{\infty}\frac{1}{a_k\ln a_k}$ diverges. $\square$