Problem 11697
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Proposed by Moubinool Omarjee (France).

Let n and q be integers, with $2n > q \geq 1$. Let

$$f(t) = \int_{\mathbb{R}^q} \frac{e^{-t(x_1^{2n}+\cdots+x_q^{2n})}}{1 + x_1^{2n} + \cdots + x_q^{2n}}\, dx_1 \cdots dx_q.$$

Prove that $\lim_{t\to+\infty} t^{q/2n} f(t) = n^{-q}(\Gamma(1/2n))^q$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First we remind the definition of the Gamma function for $x > 0$, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$. For $i = 1, \ldots, q$, let $x_i^{2n} = u_i$, then $dx_i = \frac{1}{2n}(u_i)^{\frac{1}{2n}-1}du_i$ and

$$\begin{aligned}
f(t) &= 2^q \int_{[0,+\infty)^q} \frac{e^{-t(x_1^{2n}+\cdots+x_q^{2n})}}{1 + x_1^{2n} + \cdots + x_q^{2n}}\, dx_1 \cdots dx_q \\
&= n^{-q} \int_{[0,+\infty)^q} \frac{\prod_{i=1}^q u_i^{\frac{1}{2n}-1} e^{-tu_i}}{1 + u_1 + \cdots + u_q}\, du_1 \cdots du_q \\
&= n^{-q} \int_{[0,+\infty)^q} \prod_{i=1}^q u_i^{\frac{1}{2n}-1} e^{-tu_i} \left(\int_0^\infty e^{-s(1+u_1+\cdots+u_q)}\,ds\right) du_1 \cdots du_q \\
&= n^{-q} \int_0^\infty e^{-s} \left(\int_{[0,+\infty)^q} \prod_{i=1}^q u_i^{\frac{1}{2n}-1} e^{-(s+t)u_i}\, du_1 \cdots du_q\right) ds \\
&= n^{-q} \int_0^\infty e^{-s} \left(\int_0^{+\infty} u^{\frac{1}{2n}-1} e^{-(s+t)u}\,du\right)^q ds \\
&= n^{-q} \int_0^\infty e^{-s} \left(\frac{1}{(s+t)^{1/2n}} \int_0^{+\infty} r^{\frac{1}{2n}-1} e^{-r}\,dr\right)^q ds \\
&= n^{-q} \int_0^\infty e^{-s} \left(\frac{\Gamma(1/2n)}{(s+t)^{1/2n}}\right)^q ds = n^{-q}(\Gamma(1/2n))^q \int_0^\infty \frac{e^{-s}}{(s+t)^{q/2n}}\,ds.
\end{aligned}$$

Hence

$$\lim_{t\to+\infty} t^{q/2n} f(t) = n^{-q}(\Gamma(1/2n))^q \lim_{t\to+\infty} \int_0^\infty \left(\frac{t}{s+t}\right)^{q/2n} e^{-s}\,ds = n^{-q}(\Gamma(1/2n))^q$$

because

$$0 \leq \left(\frac{t}{s+t}\right)^{q/2n} \leq 1, \quad \lim_{t\to+\infty} \left(\frac{t}{s+t}\right)^{q/2n} = 1 \quad \text{and} \quad \int_0^\infty e^{-s}\,ds = \Gamma(1) = 1.$$

It seems that the condition $2n > q$ is not necessary. □