Problem 11696
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Proposed by Enkel Hysnelaj (Australia) and Elton Bojaxhiu (Germany).

Let T be a triangle with sides of length a, b, c, inradius r, circumradius R, and semiperimeter s. Show that

$$\frac{1}{2(r^2+4Rr)}+\frac{1}{9}\sum_{cyc}\frac{1}{a(s-a)}\geq\frac{4}{9}\sum_{cyc}\frac{1}{9Rr-a(s-a)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By triangle inequality we have that $x = s-a$, $y = s-b$, and $z = s-c$ are non negative numbers and it is well known that

$$\sigma_1 = a+b+c = s, \quad \sigma_2 = ab+bc+ca = r^2+4Rr, \quad \sigma_3 = abc = sr^2.$$

Hence, $4Rr = (\sigma_1\sigma_2-\sigma_3)/\sigma_1$ and the required inequality is equivalent to

$$9+2\sigma_2\sum_{\text{cyc}}\frac{1}{(\sigma_1-x)x}\geq 32\sigma_1\sigma_2\sum_{\text{cyc}}\frac{1}{9(\sigma_1\sigma_2-\sigma_3)-4\sigma_1(\sigma_1-x)x}.$$

After expanding we obtain the following symmetric homogeneous inequality

$$\begin{aligned}450[7,5,0]+1205[7,4,1]+1733[7,3,2]+450[6,6,0]+2895[6,5,1]\\+1826[6,4,2]+237[5,5,2]\geq 345[6,3,3]+6175[5,4,3]+2276[4,4,4]\end{aligned}$$

where

$$[\alpha,\beta,\gamma]=\sum_{\text{sym}}x^\alpha y^\beta z^\gamma$$

with $\alpha\geq\beta\geq\gamma\geq 0$. Finally, this inequality holds by Muirhead's inequality because

$345[7,5,0]\geq 345[6,3,3]$,
$105[7,5,0]+1205[7,4,1]+1733[7,3,2]+450[6,6,0]+619[6,5,1]+1826[6,4,2]+237[5,5,2]\geq 6175[5,4,3]$,
$2276[6,5,1]\geq 2276[4,4,4]$.

$\square$