Problem 11692
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Proposed by Cezar Lupu (USA) and Stefan Spataru (Romania).

Let a_1, a_2, a_3, a_4 *be real numbers in* $(0,1)$ *with* $a_4 = a_1$. *Show that*

$$\frac{3}{1-a_1a_2a_3} + \sum_{k=1}^{3} \frac{1}{1-a_k^3} \geq \sum_{k=1}^{3} \left(\frac{1}{1-a_k^2a_{k+1}} + \frac{1}{1-a_ka_{k+1}^2} \right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By Schur's inequality, for any $x, y, z \geq 0$,

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2.$$

Let $x = a_1^k$, $y = a_2^k$, $z = a_3^k$ for $k \geq 0$, then

$$(a_1^3)^k + (a_2^3)^k + (a_3^3)^k + 3(a_1a_2a_3)^k \geq (a_1^2a_2)^k + (a_1a_2^2)^k + (a_2^2a_3)^k + (a_2a_3^2)^k + (a_3^2a_1)^k + (a_3a_1^2)^k.$$

Finally, since $a_1, a_2, a_3 \in (0,1)$, by summing for $k \geq 0$, the geometric series converge and easily we obtain the required inequality. $\square$