Problem 11690
(American Mathematical Monthly, Vol.120, May 2013)

Proposed by Proposed by Pal Peter Dalyay (Hungary). .

Let M be a point in the interior of a convex polygon with vertices $A_1, \dots, A_n$ in order. For $1 \le i \le n$, let r_i be the distance from M to A_i, and let R_i be the radius of the circumcircle of triangle MA_iA_{i+1}, where $A_{n+1} = A_1$. Show that

$$\sum_{i=1}^{n} \frac{R_i}{r_i + r_{i+1}} \ge \frac{n}{4\cos(\pi/n)}.$$

Solution proposed by Radouan Boukharfane, Polytechnique de Montreal, Canada, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

We first note that, by Mollweide's formula, for any non-degenerate triangle ΔABC we have that

$$\frac{c}{a+b} = \frac{\sin(C/2)}{\cos((A-B)/2)}.$$

Hence, since $|A - B| < \pi$, it follows that $0 < \cos((A - B)/2) \le 1$ and, by the law of sines,

$$\frac{R}{a+b} = \frac{c}{2\sin(C)(a+b)} = \frac{\sin(C/2)}{2\sin(C)\cos((A-B)/2)} \ge \frac{1}{4\cos(C/2)}$$

where R is the radius of the circumcircle of ΔABC.
Now, let us consider the triangle ΔMA_iA_{i+1} for $i = 1, \dots, n$, then by the above inequality

$$\sum_{i=1}^{n} \frac{R_i}{r_i + r_{i+1}} \ge \frac{1}{4}\sum_{i=1}^{n} \frac{1}{\cos(A_iMA_{i+1}/2)} \ge \frac{n}{4\cos\left(\sum_{i=1}^{n} A_iMA_{i+1}/2n\right)} = \frac{n}{4\cos(\pi/n)}$$

where we used the following facts: $1/\cos(x)$ is convex in $[0, \pi/2)$, and

$$\sum_{i=1}^{n} A_iMA_{i+1} = 2\pi$$

because M is an interior point of a convex polygon. $\square$