Problem 11686
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Proposed by Michel Bataille (France).

Let x, y, z be positive real numbers such that $x + y + z = \pi/2$. Prove that

$$\frac{\cot x + \cot y + \cot z}{\tan x + \tan y + \tan z} \geq 4(\sin^2 x + \sin^2 y + \sin^2 z).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $x = 2\alpha$, $y = 2\beta$ and $z = 2\gamma$, where α, β, γ are the angles of a triangle. with sidelengths a, b, c. Then triangle inequality implies that $u = s - a$, $v = s - b$, and $w = s - c$ are non negative numbers where $s = (a + b + c)/2$. It is well known that

$$\tan x = \sqrt{\frac{vw}{su}}, \quad \tan y = \sqrt{\frac{wu}{sv}}, \quad \tan z = \sqrt{\frac{uv}{sw}}.$$

Therefore

$$\cot x + \cot y + \cot z = \cot x \cot y \cot z = \sqrt{\frac{s^3}{uvw}},$$
$$\cot x \cot y + \cot y \cot z + \cot z \cot x = s\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)$$
$$\sin^2 x + \sin^2 y + \sin^2 z = \frac{\tan^2 x}{1 + \tan^2 x} + \frac{\tan^2 y}{1 + \tan^2 y} + \frac{\tan^2 z}{1 + \tan^2 z} = \frac{vw}{su + vw} + \frac{wu}{sv + uz} + \frac{uw}{sw + uv}.$$

Hence, the required inequality becomes

$$\frac{(u + v + w)^2}{uv + vw + wu} \geq \frac{4vw}{(u + w)(u + v)} + \frac{4wu}{(v + u)(v + w)} + \frac{4uv}{(w + v)(w + u)}$$

or the following symmetric homogeneous inequality in the variables $u, v, w \geq 0$,

$$(u - v)^2 W + (v - w)^2 U + (w - u)^2 V \geq 0$$

where

$$U = v^2(w+u)+w^2(v+u)-u^2(v+w), \quad V = w^2(u+v)+u^2(w+v)-v^2(w+u), \quad W = u^2(v+w)+v^2(u+w)-w^2(u+v).$$

Without loss of generality we assume that $u \geq v \geq w \geq 0$, then

$$V = (u - v)(uv + vw + wu) + w^2(u + v) \geq 0, \quad \text{and} \quad W = (u - w)(uv + vw + wu) + v^2(u + w) \geq 0.$$

Therefore

$$(u - v)^2 W + (v - w)^2 U + (w - u)^2 V \geq (v - w)^2(U + V) = 2(v - w)^2 w^2(u + v) \geq 0.$$

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