Problem 11685
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Proposed by Donald Knuth (USA).

Prove that

$$\prod_{k=0}^{\infty}\left(1+\frac{1}{2^{2^k}-1}\right)=\frac{1}{2}+\sum_{k=0}^{\infty}\frac{1}{\prod_{j=0}^{k-1}\left(2^{2^j}-1\right)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that $F(x)=G(x)$ for $|x|<2/3$ where

$$F(x)=\sum_{n=0}^{\infty}a_nx^n=\prod_{k=0}^{\infty}\left(\frac{1}{1-x^{2^k}}\right)\quad\text{and}\quad G(x)=\sum_{n=0}^{\infty}b_nx^n=1-x+2\sum_{k=0}^{\infty}\frac{x^{2^k}(1-x^{2^k})}{\prod_{j=0}^{k}\left(1-x^{2^j}\right)}.$$

The required equation is equivalent to $F(1/2)=G(1/2)$.
Note that

$$F(x^2)=\prod_{k=0}^{\infty}\left(\frac{1}{1-x^{2^{k+1}}}\right)=\prod_{k=1}^{\infty}\left(\frac{1}{1-x^{2^k}}\right)=(1-x)F(x)$$

which implies that $a_{2n+1}=a_{2n}$, $a_{2n}=a_{2n-1}+a_n$ for $n\geq 1$.
Moreover

$$\begin{aligned}G(x^2)&=1-x^2+2\sum_{k=0}^{\infty}\frac{x^{2^{k+1}}(1-x^{2^{k+1}})}{\prod_{j=0}^{k}\left(1-x^{2^{j+1}}\right)}=1-x^2+2\sum_{k=0}^{\infty}\frac{x^{2^{k+1}}(1-x^{2^{k+1}})}{\prod_{j=1}^{k+1}\left(1-x^{2^j}\right)}\\&=1-x^2+2(1-x)\sum_{k=0}^{\infty}\frac{x^{2^{k+1}}(1-x^{2^{k+1}})}{\prod_{j=0}^{k+1}\left(1-x^{2^j}\right)}=1-x^2+2(1-x)\sum_{k=1}^{\infty}\frac{x^{2^k}(1-x^{2^k})}{\prod_{j=0}^{k}\left(1-x^{2^j}\right)}\\&=(1-x)\left(1+x+2\left(\sum_{k=0}^{\infty}\frac{x^{2^k}(1-x^{2^k})}{\prod_{j=0}^{k}\left(1-x^{2^j}\right)}-x\right)\right)=(1-x)G(x).\end{aligned}$$

Therefore $b_{2n+1}=b_{2n}$, $b_{2n}=b_{2n-1}+b_n$ for $n\geq 1$.
Since $a_0=F(0)=1=G(0)=b_0$, it follows that $a_n=b_n$ for all $n\geq 0$, and therefore, the power series F and G coincide in $(-R,R)$ where $R=1/\limsup_{n\to\infty}(a_n)^{1/n}$. It is easy to verify by induction that $1\leq a_n\leq (3/2)^n$ for all $n\geq 0$, so $R\geq 2/3$. $\square$