Problem 11682
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Compute

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let

$$H_n(d)=\sum_{k=1}^{n}\frac{1}{k^d},\quad H_n(-d)=\sum_{k=1}^{n}\frac{(-1)^k}{k^d},\quad H_n(-1,-1)=\sum_{1\leq k<j\leq n}\frac{(-1)^{j+k}}{kj},$$
$$H_n(1,-1)=\sum_{1\leq k<j\leq n}\frac{(-1)^j}{kj},\quad H_n(-1,1)=\sum_{1\leq k<j\leq n}\frac{(-1)^k}{kj}.$$

We first note that

$$a_n=\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2=\left((-1)^n\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k}\right)^2=\left(\ln 2-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)^2.$$

So a_n is strictly decreasing to zero and the desired series converge. For any positive integer N,

$$\begin{aligned}S_{2N}&=(\ln 2)^2+\sum_{n=1}^{2N}(-1)^n a_n=(\ln 2)^2+\sum_{n=1}^{2N}(-1)^n\left(\ln 2-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\right)^2\\&=(\ln 2)^2-2(\ln 2)\sum_{n=1}^{2N}(-1)^n\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}+\sum_{n=1}^{2N}(-1)^n\sum_{k=1}^{n}\frac{1}{k^2}\\&\qquad+2\sum_{n=1}^{2N}(-1)^n\sum_{k=1}^{n}\sum_{j=k+1}^{n}\frac{(-1)^{k+j}}{kj}.\end{aligned}$$

Moreover

$$\sum_{n=1}^{2N}(-1)^n\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}=\sum_{k=1}^{2N}\frac{(-1)^{k-1}}{k}\sum_{n=k}^{2N}(-1)^n=-\frac{1}{2}H_N(1),$$
$$\sum_{n=1}^{2N}(-1)^n\sum_{k=1}^{n}\frac{1}{k^2}=\sum_{k=1}^{2N}\frac{1}{k^2}\sum_{k=n}^{2N}(-1)^n=\frac{1}{4}H_N(2)$$
$$\begin{aligned}\sum_{n=1}^{2N}(-1)^n\sum_{k=1}^{n}\sum_{j=k+1}^{n}\frac{(-1)^{k+j}}{kj}&=\sum_{k=1}^{2N}\sum_{n=k}^{2N}\sum_{j=k+1}^{n}\frac{(-1)^{n+k+j}}{kj}=\sum_{k=1}^{2N}\sum_{j=k+1}^{2N}\frac{(-1)^{k+j}}{kj}\sum_{n=j}^{2N}(-1)^n\\&=\frac{1}{2}H_{2N}(-1,1)+\frac{1}{2}H_{2N}(-1,-1).\end{aligned}$$

Hence,

$$\begin{aligned}
S_{2N} &= (\ln 2)^2 + (\ln 2)H_N(1) + \frac{1}{4}H_{2N}(2) + H_{2N}(-1,1) + H_{2N}(-1,-1) \\
&= (\ln 2)^2 + (\ln 2)H_N(1) + \frac{1}{4}H_{2N}(2) + H_{2N}(-1)H_{2N}(1) - H_{2N}(1,-1) - H_{2N}(-2) \\
&\quad + \frac{1}{2}(H_{2N}(-1))^2 - \frac{1}{2}H_{2N}(2).
\end{aligned}$$

Finally

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2 = \lim_{N\to+\infty} S_{2N} = \frac{\pi^2}{24}$$

because

$$\begin{aligned}
&H_n(1) = \ln(n) + \gamma + O(1/n), \qquad H_n(-1) = -\ln 2 + O(1/n), \\
&H_n(2) = \pi^2/6 + o(1), \qquad H_n(-2) = -\pi^2/12 + o(1), \qquad H_n(1,-1) = (\ln 2)^2/2 + o(1).
\end{aligned}$$

□