Problem 11680
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Proposed by Benjamin Bogosel (France), and Cezar Lupu (Romania).

Let $x_1, \dots, x_n$ be nonnegative real numbers. Show that

$$\left(\sum_{i=0}^{n} \frac{x_i}{i}\right)^4 \leq 2\pi^2 \sum_{i,j=1}^{n} \frac{x_i x_j}{i+j} \sum_{i,j=1}^{n} \frac{x_i x_j}{(i+j)^3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It suffices to apply the following integral version of Carlson's inequality: if f is a non negative function defined in $[0, +\infty)$ such that $f(t), tf(t) \in L^2([0, +\infty)$ then

$$\left(\int_0^{+\infty} f(t)\, dt\right)^4 \leq \pi^2 \int_0^{+\infty} (f(t))^2\, dt \int_0^{+\infty} t^2 (f(t))^2\, dt.$$

Indeed, take $f(t) = \sum_{i=1}^{n} x_i e^{-it}$, then, since for any nonnegative a

$$\int_0^{+\infty} e^{-at}\, dt = \frac{1}{a} \quad \text{and} \quad \int_0^{+\infty} t^2 e^{-at}\, dt = \frac{2}{a^3},$$

it follows that

$$\int_0^{+\infty} f(t)\, dt = \sum_{i=1}^{n} x_i \int_0^{+\infty} e^{-it}\, dt = \sum_{i=0}^{n} \frac{x_i}{i},$$

$$\int_0^{+\infty} (f(t))^2\, dt = \sum_{i,j=1}^{n} x_i x_j \int_0^{+\infty} e^{-(i+j)t}\, dt = \sum_{i=0}^{n} \frac{x_i x_j}{i+j},$$

$$\int_0^{+\infty} t^2 (f(t))^2\, dt = \sum_{i,j=1}^{n} x_i x_j \int_0^{+\infty} t^2 e^{-(i+j)t}\, dt = 2 \sum_{i=0}^{n} \frac{x_i x_j}{(i+j)^3},$$

and the required inequality holds by the above Carlson's inequality. □