Problem 11678
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Let F_k be the kth Fibonacci number, where $F_0 = 0$ and $F_1 = 1$. For $n \geq 1$ let A_n be an $(n{+}1)\times(n{+}1)$ matrix with entries $a_{j,k}$ given by $a_{0,k} = a_{k,0} = F_k$ for $0 \leq k \leq n$ and by $a_{j,k} = a_{j-1,k} + a_{j,k-1}$ for $j, k \geq 1$. Compute the determinant of A_n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let B_n be the $(n+1)\times(n+1)$ upper triangular matrix with entries

$$b_{j,k} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j-1 \text{ or } i = j-2, \\ 0 & \text{otherwise.} \end{cases}$$

then, by the definition of the numbers $a_{j,k}$, it is easy to verify that

$$B_n^t \cdot A_n \cdot B_n = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & 2P_{n-1} & \\ 0 & 0 & & & \end{array}\right]$$

where P_{n-1} is the the $(n-1)\times(n-1)$ Pascal matrix:

$$(P_{n-1})_{j,k} = \binom{i+j}{i} \quad \text{for } i = 0, \ldots, n-2 \text{ and } j = 0, \ldots, n-2.$$

Hence

$$\det(A_n) = \det(B_n^t \cdot A_n \cdot B_n) = -\det(2P_{n-1}) = -2^{n-1}$$

because $\det(M_n) = 1$, $\det(P_{n-1}) = 1$. $\square$