**Problem 11677**
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Proposed by Albert Stadler (Switzerland).

*Evaluate*

$$\prod_{n=1}^{\infty}\left(1+2e^{-n\pi\sqrt{3}}\cosh\left(\frac{n\pi}{\sqrt{3}}\right)\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\tau = i/\sqrt{3}$ and $q = \exp(2\pi i\tau)$ then

$$\begin{aligned}
\prod_{n=1}^{\infty}\left(1+2e^{-n\pi\sqrt{3}}\cosh\left(\frac{n\pi}{\sqrt{3}}\right)\right) &= \prod_{n=1}^{\infty}\left(1+q^n+q^{2n}\right)\\
&= \frac{\prod_{n=1}^{\infty}(1-q^{3n})}{\prod_{n=1}^{\infty}(1-q^n)}\\
&= \exp(-\pi i\tau/6)\frac{\eta(3\tau)}{\eta(\tau)}\\
&= \exp(-\pi i\tau/6)\frac{\eta(-1/\tau)}{\eta(\tau)}\\
&= \exp(-\pi i\tau/6)\sqrt{-i\tau}\\
&= \frac{\exp(\pi\sqrt{3}/18)}{3^{1/4}}
\end{aligned}$$

where we used the following identity which holds for $\mathrm{Im}(\tau) > 0$

$$\eta(-1/\tau) = \sqrt{-i\tau}\;\eta(\tau)$$

with

$$\eta(\tau) = \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi i n\tau)),$$

the Dedekind eta function. □