Problem 11676
(American Mathematical Monthly, Vol.119, November 2012)

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For real t, find

$$\lim_{x\to\infty} x^{\sin^2 t}\left(\Gamma(x+2)^{(\cos^2 t)/(x+1)} - \Gamma(x+1)^{(\cos^2 t)/x}\right).$$

Here, Γ is the Euler gamma function.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that for $x \to \infty$

$$\ln(\Gamma(x)) = x\ln(x) - x - \frac{1}{2}\ln(x) + \ln(\sqrt{2\pi}) + O(1/x).$$

Hence, if $n, m \geq 0$ then

$$\frac{\ln(\Gamma(x+n))}{x+m} = \ln(x) - 1 + \frac{(2(n-m)-1)\ln(x)}{2x} + \frac{\ln(\sqrt{2\pi})+m}{x} + o(1/x).$$

Let $a = \cos^2 t$ and $b = \sin^2 t$. Therefore

$$\begin{aligned}\Gamma(x+n)^{a/(x+m)} &= \exp(a\ln(\Gamma(x+n))/(x+m))\\ &= \exp(a(\ln(x) - 1 + (2(n-m)-1)\ln(x)/(2x) + (\ln(\sqrt{2\pi})+m)/x + o(1/x)))\\ &= e^{-a}x^a\exp(a((2(n-m)-1)\ln(x)/(2x) + (\ln(\sqrt{2\pi})+m)/x + o(1/x))),\\ &= e^{-a}x^a(1 + a((2(n-m)-1)\ln(x)/(2x) + (\ln(\sqrt{2\pi})+m)/x) + o(1/x))\end{aligned}$$

and, for $d \geq 0$,

$$x^b\left(\Gamma(x+n+d)^{a/(x+m+d)} - \Gamma(x+n)^{a/(x+m)}\right) = e^{-a}x^{a+b}(ad/x + o(1/x)) = dae^{-a} + o(1)$$

where we used the fact that $a + b = 1$. Finally, by letting $n = d = 1$, $m = 0$ we obtain,

$$\lim_{x\to\infty} x^{\sin^2 t}\left(\Gamma(x+2)^{(\cos^2 t)/(x+1)} - \Gamma(x+1)^{(\cos^2 t)/x}\right) = (\cos^2 t)\, e^{-\cos^2 t}.$$

□