Problem 11675
(American Mathematical Monthly, Vol.119, November 2012)

Proposed by Mircea Merca (Romania).

Let p be the Euler partition function, i.e., $p(n)$ is the number of nondecreasing lists of positive integers that sum to n. Let $p(0) = 1$, and let $p(n) = 0$ for $n < 0$. Prove that for $n \geq 0$ with $n \neq 3$,

$$p(n) - 4p(n-3) + 4p(n-5) - p(n-8) > 0.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $P(x)$ be the generating function of the integer partitions

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

We have to prove that, in the following generating function, the coefficient of x^n is positive for all $n \neq 3$:

$$\begin{aligned}
&\sum_{n=0}^{\infty}(p(n) - 4p(n-3) + 4p(n-5) - p(n-8))x^n \\
&\quad = (1 - 4x^3 + 4x^5 - x^8)P(x) \\
&\quad = (x^2(1-x)(1-x)(1-x^2) + (x^2+x+1)(1-x)(1-x^2)(1-x^3))P(x) \\
&\quad = (x^2 - x^3)\prod_{n=3}^{\infty}\frac{1}{1-x^n} + (x^2+x+1)\prod_{n=4}^{\infty}\frac{1}{1-x^n}.
\end{aligned}$$

Let $p_a(n)$ be the number of of nondecreasing lists of positive integers greater or equal to a that sum to n. Then

$$(x^2 - x^3)\prod_{n=3}^{\infty}\frac{1}{1-x^n} = \sum_{n=0}^{\infty}(p_3(n-2) - p_3(n-3))x^n = x^2 - x^3 + x^5 + x^8 + \dots$$

and the coefficient of x^n is non-negative for all $n \neq 3$. Infact $p_3(n-2) \geq p_3(n-3) \geq 1$ for all $n \geq 6$ because there is an injective map from the partitions counted by $p_3(n-3)$ to the the ones counted by $p_3(n-2)$, given by adding 1 to the greatest number in a list. Moreover

$$(x^2 + x + 1)\prod_{n=4}^{\infty}\frac{1}{1-x^n} = \sum_{n=0}^{\infty}(p_4(n-2) + p_4(n-1) + p_4(n))x^n = 1 + x + x^2 + x^4 + \dots$$

and the coefficient of x^n is positive for all $n \neq 3$ (note that $p_4(n) \geq 1$ for all $n \geq 4$). □