Problem 11672
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Proposed by Jose Luis Palacios (Venezuela).

A random walk starts at the origin and moves up-right or down-right with equal probability. What is the expected value of the first time that the walk is k steps below its then-current all time high? (Thus, for instance, with the walk $UDDUUUUDDUDD\cdots$, the walk is three steps below its maximum-so-far on step 12.)

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $a(n,k)$ be the number of walks which start at the origin and on step n are, for the first time, k steps below its maximum-so-far. Then $a(0,k)=\cdots=a(k-1,k)=0$, $a(k,k)=1$ and for $n>k$

$$a(n,k)=\sum_{j=1}^{k}\binom{k-\lfloor (j+1)/2\rfloor}{\lfloor j/2\rfloor}(-1)^{\lfloor (j-1)/2\rfloor}a(n-j,k).$$

Therefore the associated generating function is

$$F_k(x)=\sum_{n\geq 0}a(n,k)x^n=\frac{x^k}{Q_k(x)}$$

where

$$Q_k(x)=\sum_{j=0}^{k}\binom{k-\lfloor (j+1)/2\rfloor}{\lfloor j/2\rfloor}(-1)^{\lfloor (j+1)/2\rfloor}x^j.$$

Since the random walk moves up-right or down-right with equal probability, the desired expected value is given by

$$\begin{aligned}\left(\frac{d}{dx}\left(F_k(x/2)\right)\right)_{x=1} &= \frac{k(1/2)^{k-1}Q_k(1/2)-(1/2)^kQ_k'(1/2)}{2Q_k(1/2)^2}\\ &= \frac{k(1/2)^{k-1}1/2^k-(1/2)^k(-k^2/2^{k-1})}{2(1/2^k)^2}=k(k+1),\end{aligned}$$

because $Q_k(1/2)=1/2^k$ and $Q_k(1/2)=-k^2/2^{k-1}$.
The last two equalities follow from the fact that

$$\sum_{k\geq 0}Q_k(x)z^k=\frac{1-xz}{1-z+x^2z^2}$$

which implies that

$$\sum_{k\geq 0}Q_k(1/2)z^k=\frac{1}{1-z/2},\quad \sum_{k\geq 0}Q_k'(1/2)z^k=-\frac{z(z+2)}{2(1-z/2)^3}.$$

□