Problem 11670
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Proposed by Miranda Bakke, Benson Wu, and Bogdan Suceava (USA).

Prove that if $n \geq 3$ *and* $a_1, \ldots, a_n > 0$, *then*

$$\frac{(n-1)}{4} \sum_{k=1}^{n} a_k \geq \sum_{1 \leq j < k \leq n} \frac{a_j a_k}{a_j + a_k},$$

with equality if and only if all a_j *are equal.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The inequality $(a_j - a_k)^2 \geq 0$ is equivalent to

$$\frac{a_j + a_k}{4} \geq \frac{a_j a_k}{a_j + a_k}.$$

By summing over $1 \leq j < k \leq n$, we obtain

$$\frac{1}{4} \sum_{1 \leq j < k \leq n} (a_j + a_k) \geq \sum_{1 \leq j < k \leq n} \frac{a_j a_k}{a_j + a_k}$$

which yields the required inequality because

$$\frac{1}{4} \sum_{1 \leq j < k \leq n} (a_j + a_k) = (n-1) \sum_{k=1}^{n} a_k.$$

If all a_j are equal then the equality holds. On the other hand, if $a_{j_0} \neq a_{k_0}$ for some $1 \leq j_0 < k_0 \leq n$ then $(a_{j_0} - a_{k_0})^2 > 0$ and

$$\frac{a_{j_0} + a_{k_0}}{4} > \frac{a_{j_0} a_{k_0}}{a_{j_0} + a_{k_0}}.$$

This implies that in the required inequality the left-hand side is greater than right-hand side. □