Problem 11668
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Proposed by Dimitris Stathopoulos (Greece).

For positive integer n and $i \in \{0,1\}$, let $D_i(n)$ be the number of derangements on n elements whose number of cycles has the same parity as i. Prove that $D_1(n) - D_0(n) = n-1$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The (unsigned) Stirling number of the first kind $\left[{n \atop k}\right]$ counts the number of permutations of n objects with k cycles. Hence, by the inclusion-exclusion principle, the number of derangements (permutations without fixed points) of n objects with k cycles is given by the formula

$$D(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n}{j} \left[{n-j \atop k-j}\right].$$

It follows that

$$\begin{aligned}
D_1(n) - D_0(n) &= \sum_{k=0}^{\lfloor n/2 \rfloor} D(n,2k+1) - \sum_{k=1}^{\lfloor n/2 \rfloor} D(n,2k) = \sum_{k=1}^{n} (-1)^{k-1} D(n,k) \\
&= \sum_{k=0}^{n} (-1)^{k-1} \sum_{j=0}^{k} (-1)^j \binom{n}{j} \left[{n-j \atop k-j}\right] \\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^{j+k-1} \binom{n}{j} \left[{n-j \atop k-j}\right] \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n-j} (-1)^{k-1} \binom{n}{j} \left[{n-j \atop k}\right] \\
&= -\sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} (-1)^k \left[{j \atop k}\right] \\
&= -\sum_{j=0}^{1} \binom{n}{j} \sum_{k=0}^{j} (-1)^k \left[{j \atop k}\right] = -(1 + n(0-1)) = n-1.
\end{aligned}$$

where in the last step we used the fact that for $j > 1$

$$\sum_{k=0}^{j} (-1)^k \left[{j \atop k}\right] = \left(\sum_{k=0}^{j} \left[{j \atop k}\right] x^k \right)_{x=-1} = (x(x+1)\ldots(x+j-1))_{x=-1} = 0.$$

□