Problem 11667
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Proposed by Cezar Lupu (Romania).

Let f, g, and h be elements of an inner product space over $\mathbb{R}$, with $\langle f,g\rangle = 0$.
(a) Show that $\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle^2 \geq 4\langle g,h\rangle^2\langle h,f\rangle^2$.
(b) Show that $\langle f,f\rangle\langle h,h\rangle\langle h,f\rangle^2 + \langle g,g\rangle\langle h,h\rangle\langle g,h\rangle^2 \geq 4\langle g,h\rangle^2\langle h,f\rangle^2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

If $f = 0$ or $g = 0$ then (a) and (b) are trivial. Assume that $f \neq 0$ and $g \neq 0$. Let $r = h - \alpha f - \beta g$ where

$$\alpha = \frac{\langle f,h\rangle}{\|f\|^2}, \qquad \beta = \frac{\langle g,h\rangle}{\|g\|^2},$$

then $\langle r,f\rangle = \langle r,g\rangle = 0$ and $\langle h,h\rangle = \alpha^2\|f\|^2 + \beta^2\|g\|^2 + \|r\|^2$.
(a) We have that

$$\begin{aligned}\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle^2 - 4\langle g,h\rangle^2\langle h,f\rangle^2 &= \|f\|^2\|g\|^2(\alpha^2\|f\|^2 + \beta^2\|g\|^2 + \|r\|^2)^2 - 4\alpha^2\|f\|^4\beta^2\|g\|^4 \\ &\geq \|f\|^2\|g\|^2\left((\alpha^2\|f\|^2 + \beta^2\|g\|^2)^2 - 4\alpha^2\beta^2\|f\|^2\|g\|^2\right) \\ &= \|f\|^2\|g\|^2\left(\alpha^2\|f\|^2 - \beta^2\|g\|^2\right)^2 \geq 0.\end{aligned}$$

(b) We have that

$$\begin{aligned}&\langle f,f\rangle\langle h,h\rangle\langle h,f\rangle^2 + \langle g,g\rangle\langle h,h\rangle\langle g,h\rangle^2 - 4\langle g,h\rangle^2\langle h,f\rangle^2 \\ &\quad= (\alpha^2\|f\|^2 + \beta^2\|g\|^2 + \|r\|^2)(\alpha^2\|f\|^6 + \beta^2\|g\|^6) - 4\alpha^2\|f\|^4\beta^2\|g\|^4 \\ &\quad\geq (\alpha^2\|f\|^2 + \beta^2\|g\|^2)(\alpha^2\|f\|^6 + \beta^2\|g\|^6) - 4\alpha^2\beta^2\|f\|^4\|g\|^4 \\ &\quad\geq (\alpha^2\|f\|^4 + \beta^2\|g\|^4)^2 - 4\alpha^2\beta^2\|f\|^4\|g\|^4 \\ &\quad\geq (\alpha^2\|f\|^4 - \beta^2\|g\|^4)^2 \geq 0\end{aligned}$$

where at the third step we used the fact that, by Cauchy-Schwarz inequality,

$$(\alpha^2\|f\|^2 + \beta^2\|g\|^2)(\alpha^2\|f\|^6 + \beta^2\|g\|^6) \geq (\alpha^2\|f\|^4 + \beta^2\|g\|^4)^2.$$

$\square$