Problem 11663
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The unit interval is broken at two randomly chosen points along its length. Show that the probability that the lengths of the resulting three intervals are the heights of a triangle is equal to

$$\frac{12\sqrt{5}\log((3+\sqrt{5})/2)}{25} - \frac{4}{5}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $x, y, z \geq 0$ the lengths of the three intervals. Since $x+y+z=1$, it follows that (x, y) is randomly chosen inside the triangle T given by the inequalities: $x > 0$, $y > 0$, and $z = 1 - x - y > 0$ (we leave out the boundary which has null measure).
It is known that $x, y, z > 0$ are the the heights of a triangle if and only if $a = 2S/x$, $b = 2S/y$, and $c = 2S/z$ satisfy the triangle inequality (S is the area of such triangle), that is

$$\frac{1}{z} + \frac{1}{x} \geq \frac{1}{y}, \quad \frac{1}{y} + \frac{1}{z} \geq \frac{1}{x}, \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} \geq \frac{1}{z}.$$

Let p_i for $i = 1, 2, 3$ be the probability that (x, y) does not satisfy the i-th inequality. By symmetry $p_1 = p_2 = p_3$. In our case the first inequality becomes

$$y^2 - (1+x)y + (x - x^2) \geq 0$$

which is not satisfied in T if and only if

$$0 \leq y \leq \frac{1}{2}\left(1 + x - \sqrt{5x^2 - 2x + 1}\right).$$

Therefore p_1 is given by the ratio

$$\begin{aligned}
p_1 &= \frac{1}{\text{Area(T)}} \int_0^1 \frac{1}{2}\left(1 + x - \sqrt{5x^2 - 2x + 1}\right) dx \\
&= \frac{3}{2} - \frac{2}{\sqrt{5}} \int_0^1 \sqrt{\left(\frac{5x-1}{2}\right)^2 + 1}\, dx \\
&= \frac{3}{2} - \frac{4}{5\sqrt{5}} \int_{-1/2}^2 \sqrt{y^2+1}\, dy \\
&= \frac{3}{2} - \frac{2}{5\sqrt{5}} \left[y\sqrt{y^2+1} + \operatorname{arcsinh}(y)\right]_{-1/2}^2 \\
&= \frac{3}{5} - \frac{2}{5\sqrt{5}} (\operatorname{arcsinh}(2) - \operatorname{arcsinh}(-1/2)) \\
&= \frac{3}{5} - \frac{2\log((7+3\sqrt{5})/2)}{5\sqrt{5}} = \frac{3}{5} - \frac{8\sqrt{5}\log(\tau)}{25}.
\end{aligned}$$

where $\tau = (1+\sqrt{5})/2$ is the *golden mean*. Note that if the first inequality is not satisfied then $1/z + 1/x < 1/y$ implies that the other two hold:

$$\frac{1}{y} + \frac{1}{z} > \frac{1}{y} > \frac{1}{z} + \frac{1}{x} > \frac{1}{x} \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} > \frac{1}{y} > \frac{1}{z} + \frac{1}{x} > \frac{1}{z}.$$

Hence

$$p = 1 - (p_1 + p_2 + p_3) = -\frac{4}{5} + \frac{24\sqrt{5}\log(\tau)}{25} = -\frac{4}{5} + \frac{12\sqrt{5}\log((3+\sqrt{5})/2)}{25}.$$

Note that p is approximately 0.23298 that is slightly less than 0.25 which is the probability that x, y, z are the sides of a triangle. $\square$