Problem 11659
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Proposed by Albert Stadler (Switzerland).

Let x be real with $0 < x < 1$, and consider the sequence $\{a_n\}_{n\geq 0}$ given by $a_0 = 0$, $a_1 = 1$, and, for $n > 1$,

$$a_n = \frac{a_{n-1}^2}{xa_{n-2} + (1-x)a_{n-1}}.$$

Show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

By the pentagonal number theorem, it suffices to prove that $a_n = 1/\prod_{k=1}^{n-1}(1-x^k)$ for $n > 0$. It is easy to verify that this equality holds for $n = 1$ and $n = 2$. If $n > 2$ then, by the inductive hypothesis,

$$\begin{aligned} a_n &= \frac{1}{\left(\prod_{k=1}^{n-2}(1-x^k)\right)^2} \cdot \frac{1}{\frac{x}{\prod_{k=1}^{n-3}(1-x^k)} + \frac{1-x}{\prod_{k=1}^{n-2}(1-x^k)}} \\ &= \frac{1}{\prod_{k=1}^{n-2}(1-x^k)} \cdot \frac{1}{x(1-x^{n-2}) + (1-x)} \\ &= \frac{1}{\prod_{k=1}^{n-2}(1-x^k)} \cdot \frac{1}{1-x^{n-1}} = \frac{1}{\prod_{k=1}^{n-1}(1-x^k)}. \end{aligned}$$

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