Problem 11650
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Proposed by Michael Becker (USA).

Evaluate

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} e^{-(x-y)^2} \sin^2(x^2+y^2) \frac{x^2-y^2}{(x^2+y^2)^2}\, dy\, dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Passing to polar coordinates, we get

$$\begin{aligned} I &= \int_{\rho=0}^{\infty} \int_{\theta=\pi/4}^{\pi/2} e^{-\rho^2(1-\sin 2\theta)} \sin^2(\rho^2) \frac{\rho^2 \cos 2\theta}{\rho^4} \rho d\rho\, d\theta \\ &= \int_{\rho=0}^{\infty} \frac{e^{-\rho^2} \sin^2(\rho^2)}{2\rho^3} \left(\int_{\theta=\pi/4}^{\pi/2} e^{\rho^2 \sin 2\theta}\, d(\rho^2 \sin 2\theta) \right) d\rho \\ &= \int_{\rho=0}^{\infty} \frac{e^{-\rho^2} \sin^2(\rho^2)}{2\rho^3} \left[e^{\rho^2 \sin 2\theta} \right]_{\theta=\pi/4}^{\pi/2} d\rho \\ &= \int_{\rho=0}^{\infty} \frac{(e^{-\rho^2}-1) \sin^2(\rho^2)}{2\rho^3}\, d\rho \\ &= \frac{1}{4} \int_{t=0}^{\infty} \frac{(e^{-t}-1)\sin^2 t}{t^2}\, dt. \end{aligned}$$

Now it is known that (see below), for $\text{Re}(s) > 0$

$$\mathcal{L}\left[\frac{\sin^2 t}{t^2}\right](s) = \int_{t=0}^{\infty} \frac{e^{-st} \sin^2 t}{t^2} = \arctan\left(\frac{2}{s}\right) - \frac{s}{4} \log\left(\frac{s^2+4}{s^2}\right).$$

Hence

$$I = \frac{1}{4}\left(\mathcal{L}\left[\frac{\sin^2(t)}{t^2}\right](1) - \mathcal{L}\left[\frac{\sin^2(t)}{t^2}\right](0^+) \right) = \frac{1}{4} \arctan 2 - \frac{1}{16} \log 5 - \frac{\pi}{8}.$$

□

For the sake of completeness we recover here the Laplace transform we used above.
For $a \geq 0$, let

$$\varphi(a) = \mathcal{L}\left[\frac{\sin^2(at)}{t^2}\right](s)$$

then we have that

$$\varphi''(a) = \frac{d^2}{da^2}\left(\mathcal{L}\left[\frac{\sin^2(at)}{t^2}\right](s) \right) = \frac{d}{da}\left(\mathcal{L}\left[\frac{\sin(2at)}{t}\right](s) \right) = 2\mathcal{L}\left[\cos(2at)\right](s) = \frac{2s}{4a^2+s^2}.$$

Since, $\varphi(0) = \varphi'(0) = 0$, it follows that

$$\varphi'(a) = \int_0^a \frac{2s}{4a^2+s^2}\, da = \arctan\left(\frac{2a}{s}\right)$$

and

$$\varphi(a) = \int_0^a \arctan\left(\frac{2a}{s}\right) da = a \arctan\left(\frac{2a}{s}\right) - \frac{s}{4} \log\left(\frac{s^2+4a^2}{s^2}\right).$$