Problem 11649
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Proposed by Grahame Bennett (USA).

Let p be a real with $p > 1$. Let $(x_0, x_1, \dots)$ be a sequence of nonnegative real numbers. Prove that

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right)^p < \infty \Rightarrow \sum_{j=0}^{\infty}\left(\frac{1}{j+1}\sum_{k=0}^{j}x_k\right)^p < \infty.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since $2(j+1) \geq j+k+1$ for $k = 0, \dots, j$, it follows that

$$a_j = \sum_{k=0}^{\infty}\frac{x_k}{j+k+1} \geq \sum_{k=0}^{j}\frac{x_k}{j+k+1} \geq \frac{1}{2(j+1)}\sum_{k=0}^{j}x_k = \frac{b_j}{2}.$$

Hence, $\sum_{j=0}^{\infty} a_j^p < \infty$ implies that

$$\infty > 2^p\sum_{j=0}^{\infty}a_j^p \geq \sum_{j=0}^{\infty}b_j^p$$

and the proof is complete. Note that we can assume that $p > 0$. $\square$