Problem 11641
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Let f be a convex function from $\mathbb{R}$ into $\mathbb{R}$ and suppose that

$$f(x+y)+f(x-y)-2f(x)\le y^2$$

for all real x and y.
(a) Show that f is differentiable.
(b) Show that for all real x and y, $|f'(x)-f'(y)|\le|x-y|$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since f is convex, it admits left and right derivatives and for $h>0$

$$0\le\frac{f(x+h)-f(x)}{h}-\frac{f(x-h)-f(x)}{-h}=\frac{f(x+h)+f(x-h)-2f(x)}{h}\le h$$

which implies that $f'_+(x)=f'_-(x)$, that is (a).

As regards (b), by convex property of f, $f'(x)\ge f'(y)$ for $x\ge y$. Hence, it suffices to prove that $f'(x)-x\le f'(y)-y$ for $x\ge y$, which is equivalent to say that the differentiable function $F(x):=f(x)-x^2/2$ is concave. Note that

$$F(x+h)+F(x-h)-2F(x)=f(x+h)+f(x-h)-2f(x)-(x+h)^2/2-(x-h)^2/2+x^2\le h^2-h^2=0$$

So F is midpoint concave which, together with continuity, implies that F is concave. $\square$